Vol. 41 (1990), N. 3, pp. 259-276

Metroeconomica
Cappelli Editore 1990

AN ARBITRAGE INTERPRETATION OF CLASSICAL OPTIMIZATION

David P. Ellerman
EOS, Inc., Cambridge, MA 02138
(Received December 1st, 1989; final version June 13th, 1990)

ABSTRACT

The paper mathematically develops the heuristic idea that the first-order necessary conditions for a classical constrained optimization problem are equivalent to a market being arbitrage-free – with the Lagrange multipliers being the arbitrage-free market prices. The arbitrage notions start with the multiplicative Kirchhoff's Voltage Law and then generalize to matrix algebra. The basic result shows the normalized arbitrage-free «market prices» (the Lagrange multipliers) resulting from a classical constrained optimizaton problem can always be obtained as a ratio of cofactors. The machinery also gives an economic interpretation of Cramer's Rule as a competitive equilibrium condition.

1. INTRODUCTION: FINDING MARKETS IN THE MATH

The purpose of this paper is to provide some of the mathematics behind the old intuitive idea that the first-order necessary conditions for a constrained classical optimization problem (equality constraints) are, in some sense, equivalent to a market being arbitrage-free – with the Lagrange multipliers corresponding to the arbitrage-free market prices.

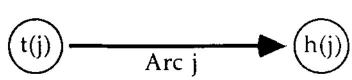
The basic mathematical result shows that the normalized arbitrage-free «market prices», i.e., the Lagrange multipliers, can always be obtained as the ratio of cofactors drawn from the matrix of first partial derivatives of the constraint and of the objective function. An amusing by-product is an economic interpretation of Cramer's Rule as a competitive equilibrium condition.

The traditional notion of market equilibrium as the equating of supply and demand is replaced with the notion of equilibrium as freedom from profitable arbitrage (e.g., as in input-output theory). The mathematics is initially developed in a graph-theoretic framework where the arbitrage-free condition is the multiplicative version of Kirchhoff's Voltage Law. Then the arbitrage notions are generalized to matrix algebra using incidence matrices as the bridge. Thus we «find markets in the math» of matrix algebra and develop an economic interpretation of cofactors, determinants, inverse matrices, and Cramer's Rule.

Arbitrage-related concepts have been applied successfully in financial economics. Merton H. Miller and Franco Modigliani used impressive arbitrage arguments in proving their famous irrelevance theorem (1958). Stephen A. Ross (Ross, 1975a, 1976b) and his colleagues have developed Arbitrage Pricing Theory so that it is now recognized as a fundamental principle in finance theory (Varian, 1987). Our purpose here is not to use arbitrage concepts to study financial markets but to «find the markets in the math» of all classical constrained optimization problems.

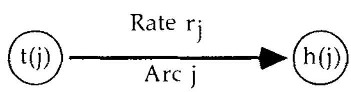
2. ARBITRAGE IN GRAPH THEORY

A *directed graph* $G = (G_0, G_1, t, h)$ is given by a set G_0 of *nodes* (numbered 0, 1, ..., m), a set G_1 of *arcs* (numbered 1, 2, ..., b), and *head* and *tail functions* $h, t: G_1 \to G_0$ which indicate that arc j is directed from its tail, the $t(j)$ node, to its head, the $h(j)$ node.



It is assumed that there are no loops at a node, i.e., $h(j) \neq t(j)$ for all arcs j. A *path from node* i to node i' is given by a sequence of arcs connected at their heads or tails that reach from node i to node i'. A graph is *connected* if there is a path between any two nodes. It is assumed that the graph G is connected. A closed circular path where no arc occurs more than once is a *cycle* [for more graph theory, see any text such as Berge and Ghouila-Houri, 1965].

Let T be any group (not necessarily commutative) written multiplicatively (i.e., a set with a binary product operation defined on it, with an identity element 1 and with every element having a multiplicative inverse or reciprocal). For most of our purposes, T can be taken as R^*, the multiplicative group of non-zero reals. In the motivating economic interpretation, a different commodity is associated with each node, and the arcs represent channels of exchange or transformation between the commodities at the nodes. A function $r: G_1 \rightarrow T$ is a *rate system* giving exchange or transformation rates. Given an arc j, one unit of the $t(j)$ commodity can be transformed into $rj = r_j$ units of the $h(j)$ commodity.



Transformation Rate r_j on Arc j

A graph (G, r) with a rate system r represents a market so it will be called a *market graph*. These group-labelled graphs are also called «voltage graphs» (Gross, 1974) or «group graphs» (Harary *et al.*, 1982).

All transformations are reversible. If arc j is traversed against the arrow, the transformation rate is the reciprocal $1/r_j$. Given a path c from node i to i', the *composite rate* $r[c]$ is the product of the rates along the path using the reciprocal rate for any arc traversed against the direction of the arrow. A function $P: G_0 \rightarrow T$ labelling the nodes is a *price system* (or absolute price system). A rate system $Q(P): G_1 \rightarrow T$ can be *derived* from a price system by taking the price ratios

$$Q(P)(j) = P(h(j))^{-1} P(t(j)).$$

Derived rate systems have certain special properties:

1) for any path c from i to i', $Q(P)[c] = P(i')^{-1} P(i)$,
2) for any two paths c and c' from i to i', $Q(P)[c] = Q(P)[c']$, and
3) for any cycle c, $Q(P)[c] = 1$.

Given a market graph (G, r), the rate system r is said to be *path-independent* if for any two paths c and c' between the same nodes, $r[c] = r[c']$. The rate system is said to be *arbitrage-free* if for any cycle c, $r[c] = 1$ («arbitrage-free» = «balanced» in much of the graph-theoretic literature following Harray, 1953).

In an idealized international currency exchange market with no transaction costs, if the product of the exchange rates around a circle is greater than one, profitable arbitrage is possible. If the product is less than one, then exchange around the circle in the opposite direction would be profitable arbitrage. Hence the market is arbitrage-free if the product of exchange rates around the circle is one.

A rate system derived from a price system has both the properties of being path-independent and arbitrage-free, and, in fact, the three properties are equivalent. That equivalence theorem is the finite multiplicative version of the calculus theorem about the equivalence of the conditions:

1) a vector field is the gradient of a potential function,

2) a line integral of the vector field between two points is path-independent, and

3) a line integral of the vector field around any closed path is zero.

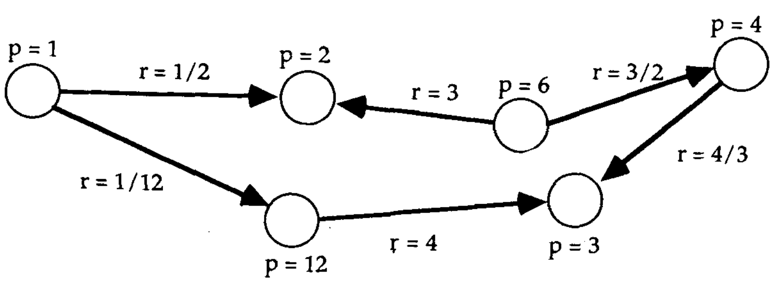
Cournot-Kirchhoff Arbitrage Theorem: Let (G, r) be a market graph with $r: G_1 \rightarrow T$ taking values in any group T. The following conditions are equivalent:

1) there exists a price system P such that $Q(P) = r$,

2) the rate system r is path-independent, and

3) the rate system r is arbitrage-free.

For a proof of this straightforward non-commutative generalization of Kirchhoff's Voltage Law (1847) and Cournot's earlier (1838) arbitrage- free condition (see 1897), see Ellerman (1984).



$r[c] = (1/2)(1/3)(3/2)(4/3)(1/4)(12) = 1$

An arbitrage-free market graph

The value group T will now be specialized to R^*, the multiplicative group of non-zero real numbers. But price system P will now be *extended* by allowing zero values in the reals R, i.e.,

$$P: G_0 \rightarrow R.$$

An extended price system P and a rate system r are *associated* if for any arc j,

$$P(h(j))r_j - P(t(j)) = 0.$$

If the price system has all non-zero values, this is the same as the rate system being derived from the price system.

The zero price system (all zero prices) is trivially associated with any rate system. If a rate system is not arbitrage-free, then the zero price system is the only associated price system. With that fixed rate system, profitable arbitrage means «getting something for nothing», so all commodities become free goods and have zero prices.

It is useful to reformulate some of the graph-theoretic notions using incidence matrices. Given (G, r), the *node-arc incidence matrix* $S = [S_{ij}]$ is the $(m + 1) \times b$ matrix where:

$$S_{ij} = \begin{cases} +r_j & \text{if} \xrightarrow{\text{Arc } j} \langle \text{Node } i \rangle \\ -1 & \text{if} \xleftarrow{\text{Arc } j} \langle \text{Node } i \rangle \\ 0 & \text{Otherwise.} \end{cases}$$

The jth column of S has a -1 and an r_j which are the results of transforming one unit of the $t(j)$ good into r_j units of the $h(j)$ good. Any linear combination of the columns would represent a possible market exchange vector using the rate system r. The negative components represent the goods given up in exchange for the goods represented by the positive components. Thus the vector space of all linear combinations of columns of S, the *column space* $\mathrm{Col}(S)$, will be called the *exchange space* of the market graph (G, r).

Let S_0, called the *reduced incidence matrix*, be the $m \times b$ matrix obtained from S by deleting the top row, the row corresponding to node 0. If G is a connected graph (a path between any two nodes), then the reduced incidence matrix S_0 has linearly independent rows, i.e., S_0 has full row rank. For let $P^* = (P_1, ..., P_m)$ be a row vector such that $P^*S_0 = 0$. Some node i was connected to the «deleted» node 0 by some arc j. In order for P^* to zero the jth column of S_0, P_i must be zero. If arc j is from node i to i' both in the node set $\{i, ..., m\}$, then $P^*S_0 = 0$ implies $P_{i'}r_j - P_i = 0$ so $P_{i'}$ and P_i are both zero or both non-zero. Thus each node connected to node i must have a zero price. Since G is connected, all prices must be zero, i.e., $P^* = 0$, so the rows of S_0 are linearly independent.

Adding back the top row, the row rank of S is either m or $m + 1$, so the column rank, i.e., the dimension of Col(S), is also either m or $m + 1$. A subspace of R^{m+1} of dimension m (one less than the dimension of the full space) is a *hyperplane* through the origin. Thus the exchange space is either a hyperplane in R^{m+1} or is the full space.

The *left nullspace* LeftNull(S) of any matrix S is the space of vectors P such that $PS = 0$. If S is the incidence matrix of a market graph (G, r) and $P = (P_0, P_1, ..., P_m)$ is in LeftNull(S), i.e., $PS = 0$, then for all arcs j

$$P_{h(j)}r_j - P_{t(j)} = 0$$

so P is a price system associated with the rate system r. Hence LeftNull(S) is called the *price space* associated with the exchange space Col(S) and the elements P are called *price vectors*. The exchange space Col(S) and the price space LeftNull(S) are *orthogonal complements* of one another, i.e.,

a) X is an exchange if and only if for any price vector P, $PX = 0$,

and

b) P is a price vector if and only if for any exchange X, $PX = 0$.

Since they are orthogonal complements, dim[Col(S)] + dim[LeftNull(S)] $= m + 1$. Since the exchange space is of dimension m or $m + 1$ (G is assumed connected), the dimension of the price space is either one or zero. A price vector with any non-zero components must have all non-zero components. Any two non-zero price vectors must be scalar multiples on one another. The two cases of a one or zero dimensional price space correspond to the cases of (G, r) being arbitrage-free or allowing profitable arbitrage. If profitable arbitrage is possible, then the fixed non-zero exchange rates r would allow one to generate any quantities of the goods so all commodities are free goods, i.e., $P = 0$ is the only price vector. These results and some easy consequences are collected together in the following theorem.

Arbitrage Theorem for Market Graphs: Let (G, r) be a market graph where G is connected and $r: G_1 \to R^*$. The following conditions are equivalent:

1) there exists a price system $P: G_0 \to R^*$ such that $Q(P) = r$,

2) the rate system r is path-independent,

3) the rate system r is arbitrage-free,

4) the price space LeftNull(S) is one-dimensional,

5) the exchange space Col(S) is a hyperplane (with a non-zero price vector as a normal vector),

6) the top row of S, s_0, can be expressed as a linear combination of the bottom m rows S_0 of S, i.e., there exist $\mu = (\mu_1, ..., \mu_2)$ such that $s_0 + \mu S_0 = 0$, and

7) if an exchange vector $b = Sx$ has $b_1 = ... = b_m = 0$, then $b_0 = 0$.

The incidence matrix treatment of market graphs suggests a generalization of the economic interpretation to a more general matrix context. The rows represent commodities. The columns specify exchange or production possibilities. Negative entries represent goods given up in exchange or inputs to production, while positive components stand for goods received or the outputs. Any scalar multiple, positive or negative, of a column also represents a possible exchange or transformation so the column space is the space of possible exchanges or transformations. The orthogonally complementary left nullspace is the set of price vectors such that all the exchanges can be obtained as trades at those prices [for more linear algebra, see any text such as Strang, 1980].

3. AN ECONOMIC INTERPRETATION OF COFACTORS, DETERMINANTS, AND CRAMER'S RULE

Let A be a square $(m+1) \times (m+1)$ matrix of reals, and let $A(k)$ be the $(m+1) \times m$ matrix obtained by deleting column k for $k = 0$, 1, ..., m. The column space Col$(A(k))$ is the space of exchanges spanned by the remaining m columns. Let

$$P(k) = (P_0(k),\ P_1(k),\ ...,\ P_m(k))$$

be the cofactors of the deleted column k. By the property of «expansion by alien cofactors», $P(k)A(k) = 0$ so $P(k)$ is a «price vector» in LeftNull$(A(k))$. The cofactors in $P(k)$ will be called the *k-prices*. The cofactors of any column of A are prices so that the exchanges defined by the remaining columns can be obtained at those market prices.

Now introduce the exchange (or productive) possibilities given by the deleted column k into the market. Its value at the reigning prices $P(k)$ is the determinant $|A|$ obtained by the cofactor expansion of column k. If $|A| \neq 0$ then any vector b can be obtained as an exchange vector $Ax = b$. As in a market which allows profitable arbitrage at fixed exchange rates, any exchange is allowed and the only price vector is the zero vector.

It is therefore desirable to temporarily alter the interpretation of the columns of A. Previously the columns represented exchange or production possibilities with **all** commodities involved as inputs or outputs listed as components. We now interpret each column as representing the (reversible!) input-output vector of a machine operating at unit level. But the machine's services are **not** represented in the input-output vector so the value of the vector can now be interpreted as the competitive rent imputed to a unit of the machine services.

The vector of cofactor k-prices $P(k) = (P_0(k), P_1(k), \ldots, P_m(k))$ can now be interpreted as a set of commodity prices which impute zero rents to all the **other** m machines (excluding the k^{th} machine). The *determinant* $|A|$ is the competitive rent (or subsidy, if negative) imputed to the unit services of machine k at those k-prices. Dividing by the determinants-as-rent, the *normalized k-prices*

$$P^*(k) = P(k)/|A|$$

are the k-prices expressed in terms of the units of machine k services as numeraire. At the normalized k-prices $P^*(k)$, all machines have zero imputed rent–save machine k which has an imputed rent of unity. This yields an economic interpretation of the *inverse matrix* A^{-1} as the *normalized price matrix*

$$P^* = \begin{bmatrix} P^*(0) \\ P^*(1) \\ \vdots \\ P^*(m) \end{bmatrix} = A^{-1}$$

obtained as the column of row vectors $P^*(k)$ for $k = 0, 1, \ldots, m$.

Suppose the machines are operated at the level $x = (x_0, x_1, \ldots, x_m)T$ so the net product vector is $Ax = b$. In *competitive equilibrium*, the competitive rents due on the machines must equal the value of the net product vector leaving no pure profits for arbitrageurs. Given a commodity price vector $P = (P_0, P_1, \ldots, P_m)$, the unit machine rents $R = (R_0, R_1, \ldots, R_m)$ must be such that the total rent Rx equals the value Pb of any net product $b = Ax$, i.e.,

$$Rx = Pb = PAx \text{ for any } x.$$

Thus competitive equilibrium requires the competitive rents $R = PA$ in terms of P.

Now consider the specific price vector $P^*(k)$. The competitive rents $R = P^*(k)A$ impute a rent only to machine k, and that rent is unity. Hence the total rent $Rx = x_k = P^*(k)Ax = P^*(k)b$ is the level of operation x_k of machine k so we have derived

Cramer's Rule as a Competitive Equilibrium Condition
Competitive Machine Rent $= x_k = P^*(k)b =$ Value of Net Product.

4. ARBITRAGE-FREE MARKET MATRICES

We now return to the «full-disclosure» interpretation of the columns of A. All commodities and services involved in the exchange or productive transformation are exposed as components of the column vectors.

When is a matrix like a market? One answer is when it is like the node-arc incidence matrix of a market graph. Let A be a rectangular $(m + 1) \times n$ matrix with $m + 1 \leq n$. Any matrix or its transpose has that form. Such a matrix A is a *market matrix* if $\text{rank}(A) \geq m$. A market matrix has a rank of m or $m + 1$. A market matrix A is said to be *arbitrage-free* if $\text{rank}(A) = m$. The node-arc incidence matrix of a connected market graph is a market matrix. The market graph is arbitrage-free (as a graph) if and only if its incidence matrix is arbitrage-free (as a matrix).

A market matrix has m linearly rows which, for notational convenience, we may take to be the bottom m rows numbered $i = 1, \ldots, m$ (the top row is row 0). Every set of m columns from the $(m + 1) \times n$ matrix A determine a $(m + 1) \times m$ submatrix A^* (taking the columns in the same order as in A). As a visual aid, we can consider a $(m + 1) \times 1$ «dummy» column vector $[?, ?, \ldots, ?]^T$ appended to the left of A^* to form a $m + 1$ square matrix. The cofactors $P_0, P_1, \ldots, P_m$ of the dummy column are the *local cofactor prices* determined by the m columns of A^*. The binomial coefficient

$$C(n, m) = n!/(m!(n - m)!)$$

gives the number of ways of choosing m columns from among n columns, so there are $C(n, m)$ vectors of local cofactor prices (not necessarily all distinct).

At least one vector of local cofactor prices is non-zero since $\operatorname{rank}(A) \leq m$. The rows have been arranged so the bottom m rows are linearly independent. Let A^* be a submatrix of m linearly independent columns so it has a vector of local cofactor prices $P^* = (P_0^*, P_1^*, \ldots, P_m^*)$ such that $P_0^* \neq 0$. These cofactor prices may be normalized by taking commodity 0 as the numeraire to obtain the relative prices:

$$(1, \mu_1, \ldots, \mu_m) = (1, P_1^*/P_0^*, \ldots, P_m^*/P_0^*).$$

To complete the development of a «market» in the market matrix A, we need to define transformation rates between commodities. The important rates are the transformation rates r_i of good i into good 0 for $i = 1, \ldots, m$ which can be defined using any m linearly independent columns A^*. The m activities are to be run at levels so that exactly one unit of good i is used-up and zero units of good j are produced or used-up for $j \neq i$, 0. Then the number of units of good 0 produced gives the transformation rate r_i so that the 1 unit of good i used-up is transformed into r_i units of good 0.

In matrix notation, let A_0^* be the bottom m rows of a $(m+1) \times m$ matrix A^* of m linearly independent columns of A so that

$$|A_0^*| = P_0^* \neq 0.$$

Let a_0^* be the top row of A^*. The activity vector x which uses-up exactly one unit of good i is the x such that

$$A_0^* x = (0, \ldots, 0, -1, 0, \ldots, 0)^T = -I_i$$

where I_i is the ith column of the $m \times m$ identity matrix I so $x = (A_0^*)^{-1} I_i$. Let

$$r_i = -a_0^* (A_0^*)^{-1} I_i$$

so the vector $r = (r_1, \ldots, r_m)$ of the transformation rates defined by A^* is

$$r = -a_0^* (A_0^*)^{-1}.$$

Cofactor Price Theorem: Given any $(m + 1) \times m$ submatrix A^* of linearly independent columns, the transformation rates r determined by A^* are equal to the normalized cofactor prices:

$$(r_1, \ldots, r_m) = (P_1^*/P_0^*, \ldots, P_m^*/P_0^*) = (\mu_1, \ldots, \mu_m).$$

Proof: For notational simplicity, we take the m columns of A^* to be the first m columns of A. The transformation rates r solve the linear equations:

$$(r_1, \ldots, r_n)\begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mm} \end{bmatrix} = rA_0^* = -a_0^* = -(a_{01}, \ldots, a_{0m}).$$

The local cofactor prices determined by A^* are the cofactors of the dummy column in the matrix:

$$\begin{bmatrix} ? & a_{01} & \ldots & a_{0m} \\ ? & a_{11} & \ldots & a_{1m} \\ \ldots & \ldots & \ldots & \ldots \\ ? & a_{m1} & \ldots & a_{mm} \end{bmatrix}.$$

Using the row form of Cramer's Rule to solve for r_1 yields:

$$r_1 = \frac{\begin{vmatrix} -a_{01} & \ldots & -a_{0m} \\ a_{21} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & \ldots & a_{1m} \\ a_{21} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mm} \end{vmatrix}} = \frac{(-1)^{2+1}\begin{vmatrix} a_{01} & \ldots & a_{0m} \\ a_{21} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mm} \end{vmatrix}}{P_0^*} = \frac{P_0^*}{P_0^*}.$$

To compute r_i, the right-hand side constants $-a_0^*$ are substituted for the ith row of A_0^* (in the numerator of the row form of Cramer's Rule). Then $i - 1$ row swaps are required to bring the $-a_0^*$ row up to the top. Factoring out the -1 leaves a $(-1)^i$ sign on the minor

$$\begin{vmatrix} a_{01} & \cdots & a_{0m} \\ a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{i-11} & \cdots & a_{i-1m} \\ a_{i+11} & \cdots & a_{i+1m} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} \end{vmatrix}$$

but P_i^* is $(-1)^{(i+1)+1} = (-1)^i$ times the same minor so $r_i = P_i^*/P_0^*$. ∎

The next theorem states a number of conditions equivalent to the market matrix A being arbirtrage-free. An arbitrage-free market has unique relative prices so the $C(n, m)$ local cofactor prices must mesh or fit together in the sense of being scalar multiples of the non-zero price vector P^* which was normalized to $(1, \mu_1, ..., \mu_m)$. The space spanned by the $C(n, m)$ cofactor price vectors is the one-dimensional space LeftNull(A). In the application to classical optimization, the μ_i's are the Lagrange multipliers of m constraints, which are thus interpreted as the unique prices of m resources in terms of the maximand as numeraire.

Arbitrage Theorem for Market Matrices: Let A be any $(m+1) \times n$ market matrix where we assume the rows 1 through m are linearly independent. Let a_0 be the top row, and let A_0 be the bottom m rows of A. The following conditions are equivalent:

1) A is arbitrage-free,

2) the price space LefNull(A) is one-dimensional,

3) the exchange space Col(A) is a hyperplane (with a cofactor price vector as a normal vector),

4) there exists $\mu = (\mu_1, ..., \mu_m)$ such that $a_0 + \mu A_0 = 0$,

5) if an exchange vector $b = Ax$ has $b_1 = ... = b_m = 0$, then $b_0 = 0$, and

Proof:

1)$\Leftrightarrow$2)$\Leftrightarrow$3) is straightforward since

$$\dim(\text{LeftNull}(A)) + \dim(\text{Col}(A)) = m + 1.$$

As noted in the paragraphs preceding the theorem, there is a

$(m+1) \times m$ submatrix A^* and a row of cofactors price $P^* = (P_0^*, P_1^*, ..., P_m^*)$ such that $P^*A^* = 0$ with $P_0^* \neq 0$. When A is arbitrage-free, the rows of A are linearly dependent so there is a price vector $P' = (P'_0, P'_1, ..., P'_m)$ such that $P'A = 0$ and thus $P'A^* = 0$. But the cofactors P^* and the vector P' can differ only by a scalar multiple, else

$$[P' - (P'_0/P_0^*)P^*]A^* = 0$$

implying the bottom m rows of A^* are linearly dependent. Hence the cofactors P^* also form a price vector in LeftNull(A) which is normal to the hyperplane Col(A).

1), ... 3)$\Leftrightarrow$4). Use $\mu_i = P_i^*/P_0^*$ for $i = 1, ..., m$.

1), ..., 3)$\Leftrightarrow$5). The dimensions of the orthogonal complements Row(A) and RightNull(A) sum to n, and similarly for Row(A_0) and RightNull(A_0). Then

1)$\Leftrightarrow$Rank(A) = Rank(A_0)
$\Leftrightarrow$dim(Row(A)) = dim(Row(A))
$\Leftrightarrow$dim(RightNull(A)) = dim(RightNull(A_0))
$\Leftrightarrow$RightNull(A) = RightNull(A_0) [RightNull(A) is a subspace of RightNull(A_0)]
$\Leftrightarrow$5). ∎

5. FIRST-ORDER NECESSARY CONDITIONS AS ARBITRAGE-FREE CONDITIONS

The intuitive arbitrage reasoning as well as the formal results for arbitrage-free market matrices can be applied to yield the first-order necessary conditions for regular constrained optimization problems with equality constraints.

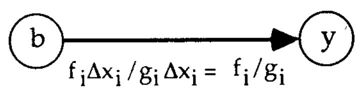
Consider the one-constraint problem:

Maximize $y = f(x_1, ..., x_n)$

Subject to: $g(x_1, ..., x_n) = b$

where all functions are continuously twice differentiable. There are two commodities, the resource b and the maximand y. There are n «instruments» with the levels of operation $x_1, ..., x_n$. At the levels $x_1, ..., x_n$, the amount of the resource used-up is $g(x_1, ..., x_n)$, and $f(x_1, ..., x_n)$ is the amount of the maximand produced.

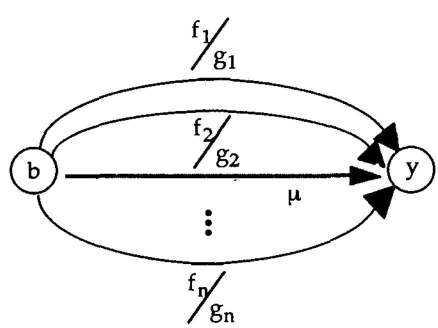
Let $x^0 = (x_1^0, ..., x_n^0)$ be levels of the instruments which use-up all of the available resource, i.e., $g(x_1^0, ..., x_n^0) = b$. Moreover, we assume that x^0 is «regular» in the sense that not all the partials $\partial g(x^0)/\partial x_i = g_i$ are zero. We consider an intuitive «marginal market» defined by the possible marginal transformations of b into y. In an international currency market (without transaction costs), there might be n banks or exchange houses which to prevent arbitrage would have to offer the same rate of exchange between any two currencies. In our market, the n instrument variables offer n ways to transform the resource b into the maximand y. A marginal variation Δx_i uses-up $g_i \Delta x_i$ units of b and produces $f_i \Delta x_i$ units of y so the rate of transformation is



The market is arbitrage-free if and only if the n transformation rates f_i/g_i provided by the n instruments are equal:

$$\frac{f_i}{g_i} = \frac{f_2}{g_2} = ... = \frac{f_n}{g_n} = \mu$$

where the common rate of transformation is the Lagrange multiplier μ.



Arbitrage diagram for the marginal market

Thus the first-order necessary conditions for x^0 to be a constrained maximum are equivalnet to the intuitive market being arbitrage-free.

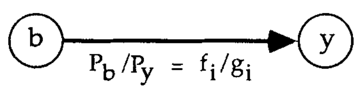
To use the machinery of market matrices, let

$$A = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ -g_1 & -g_2 & \cdots & -g_n \end{bmatrix}$$

where $-g_i$ is used instead of $+g_i$ since $g(x_1, ..., x_n)$ represents the amount of the resource used-up. Consider any column of this market matrix coupled with the dummy column to form a square matrix:

$$\begin{bmatrix} ? & f_i \\ ? & -g_i \end{bmatrix}.$$

The cofactors of the dummy column are the local prices $P_y = -g_i$ and $P_b = -f_i$ so (assuming $g_i \neq 0$) the price of b in terms of the numeraire y is the transformation rate



defined by the marginal variations in the instrument x_i. Since $m = 1$, there are $C(n, 1) = n$ sets of cofactor prices. The market matrix is arbitrage-free if and only if the n cofactor price vectors define the same price of b in terms of y.

Consider a problem with $m = 2$ constraints:

Maximize $y = f(x_1, ..., x_n)$

Subject to: $g^1(x_1, ..., x_n) = b_1$
$g^2(x_1, ..., x_n) = b_2$

where $n > m = 2$. Let G be the matrix of partials of the constraints evaluated at x^0:

$$G = \begin{bmatrix} g_1^1 & g_2^1 & \cdots & g_n^1 \\ g_1^2 & g_2^2 & \cdots & g_n^2 \end{bmatrix}.$$

The candidate point x^0 is assumed to be *regular* in the sense that G is of full row rank.

There are three commodities in the intuitive market for the problem: the maximand y and the two resources b_1 and b_2. To define a transformation rate from b_1 into y, one cannot just vary one instrument x_1 because that may also vary b_2. One must consider variations in (m) two variables x_i and x_j which leave b_2 constant and yield variations $-db_1$ and dy to define a transformation rate $r_1 = dy/db_1$ from b_1 into y. Similarly the rate for transforming b_2 into y can be defined. Using the cofactor price theorem, these rates can be obtained as ratios of local cofactor prices.

Since G is of full row rank, there are $m = 2$ instruments x_i and x_j such that

$$G^* = \begin{bmatrix} g_i^1 & g_j^1 \\ g_i^2 & g_j^2 \end{bmatrix}$$

is non-singular. Given the matrix (with the unknown dummy column)

$$\begin{bmatrix} ? & f_i & f_j \\ ? & -g_i^1 & -g_j^1 \\ ? & -g_1^2 & -g_1^2 \end{bmatrix}$$

the cofactors of the dummy column yield the prices:

$$\begin{aligned} P_y &= g_i^1 g_j^2 - g_i^2 g_j^1 \\ P_{b_1} &= f_i g_j^2 - f_j g_1^2 \\ P_{b_2} &= f_j g_1^1 - f_i g_j^1 \end{aligned}$$

where $P_y \neq 0$ by the choice of i and j. By the cofactor price theorem, the cofactor price ratios yield the transformation rates from the resources into the maximand. For instance, if x_i and x_j are varied to hold b_2 constant, the relative cofactor price of b_1 in terms of y, $P_{b_1}/P_y = \mu_1$, gives the rate of transformation of b_1 into y defined by the variation in x_i and x_j.

For the intuitive market to be arbitrage-free, all the local cofactor prices $(P'_y, P'_{b_1}, P'_{b_2})$ defined by any set of $m = 2$ instruments must be scalar multiples of the non-zero price vector (P_y, P_{b_1}, P_{b_2}). In formal terms, the market matrix defined by the problem is

$$A = \begin{bmatrix} \nabla f \\ -G \end{bmatrix}.$$

The first order necessary conditions for the candidate point to be a constrained maximum are then expressed by the market matrix A being arbitrage-free and by the other equivalent conditions given in the Arbitrage Theorem for Market Matrices.

All these results for $m = 2$ extend to the general problem with m constraints and n variables ($n > m$):

$$\begin{aligned} &\text{Maximize} \quad y = f(x_1, \ldots, x_n) \\ &\text{Subject to:} \; g^1(x_1, \ldots, x_n) = b_1 \\ &\qquad \ldots \\ &\qquad g^m(x_1, \ldots, x_n) = b_m. \end{aligned}$$

The candidate point x^0 satisfies the constraints and is *regular* in the sense that the $m \times n$ matrix $G = [g_i^j]$ is of full row rank. Thus there are m columns forming a non-singular submatrix G^*. If f^* is the vector of the corresponding m partials of f, then consider the $(m + 1) \times (m + 1)$ matrix:

$$\left[\begin{array}{c|c} ? & f^* \\ \hline ? & -G^* \end{array}\right].$$

The cofactors of the dummy column form the local cofactor prices P_y, P_{b_1}, ..., P_{b_m} determined by the m chosen instruments. The intuitive market is arbitrage-free if all the $C(m, n)$ vectors of local cofactor prices are scalar multiples of this non-zero vector. In formal terms, the first order necessary condition for the candidate point x^0 to be a constrained maximum is equivalent to the condition that the market matrix of the problem

$$A = \begin{bmatrix} \nabla f \\ -G \end{bmatrix}$$

is arbitrage-free which, in turn, is equivalent to the other conditions stated in the Arbitrage Theorem for Market Matrices.

REFERENCES

Berge C. and Ghouila-Houri A.: *Programming, Games and Transportation Networks*, John Wiley and Sons, New York, 1965.

Cournot A.: *Mathematical Principles of the Theory of Wealth*, Translated by Nathaniel Bloom, Macmillan, New York (Orig. 1838), 1987.

Ellerman David P.: «Arbitrage Theory: A Mathematical Introduction», *SIAM Review.*, 26, 241-261, 1984.

Gross J.: «Voltage graphs», *Discrete Math.* 9, 239-246, 1974.

Harary F.: «On the notion of balance of a signed graph», *Michigan Math. J.*, 2, 143-146, 1953.

Harary F., Lindstrom B. and Zetterstrom H.: «On balance in group graphs», *Networks.*, 12, 317-321, 1982.

Kirchhoff G.: «Über die Auflosung der Gleichungen, auf welche man dei der Untersuchung der linearen Verteilung galvanischer Strome gefuhrt wird», *Annalen der Physik und Chemie.*, 72, 497-508, 1847.

Modigliani F. and Miller Merton H.: «The Cost of Capital, Corporation Finance, and the Theory of Investment», *American Economic Review*, 48, 261-97, 1958.

Ross Stephen A.: «The Arbitrage Theory of Capital Asset Pricing», *J. Econ. Theory*, 13, 341-360, 1976.

Ross Stephen A.: «Return, Risk and Arbitrage», in: Friend, Irwin and Bicksler, James L. (eds.): *Risk and Return in Finance.*, Ballinger, Cambridge MA., 1976.

Strang G.: *Linear Algebra and its Application*, Second Edition., Academic Press, New York, 1980.

Varian Hal R.: «The Arbitrage Principle in Financial Economics», *The Journal of Economic Perspectives*, Vol. 1, No. 2, 55-72, 1987.

Address of the author:
Dr. David P. Ellerman
c/o GEA Ventures
Tržaška 132
61111 Ljubljana
Yugoslavia