On Abstraction in Mathematics and Indefiniteness in Quantum Mechanics

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Abstract

Given a property $S()$ on the elements of a set U, there are two notions of abstraction. The #1 notion of abstraction is to the set S of elements with the property (like an equivalence class of parallel lines); the #2 notion is an abstract entity u_S that is definite on what is common to the elements of S but is otherwise indefinite on the differences between those elements (like the abstract "direction" of the lines). The paper shows how both the #2 notion of a 'paradigm' and the #1 notion of a set may be differently modeled using incidence matrices in Boolean logic and using density matrices in probability theory. This is then used to illuminate and interpret the very similar density matrix treatment of the indefinite superposition states in quantum mechanics.

Contents

1 Introduction

The purpose of this paper is to draw out some intriguing and possibly illuminating analogies between abstraction in the philosophy of mathematics and the notion of indefiniteness in the interpretation of quantum mechanics (QM). A well-known example of an abstraction principle is Frege's "direction principle" which Stewart Shapiro described as: for any lines l_1 and l_2 in some domain, the "direction of l_1 is identical to the direction of l_2 if and only if l_1 is parallel to l_2." [19, p. 107] Abstraction turns equivalence into identity. But there are two different ways for this abstraction principle to be satisfied. The version often used by the proverbial 'working mathematician' will be called the #1 abstraction, namely, just the equivalence class. If $[l]$ is the parallelism equivalence class of the line l, then the direction principle is clearly satisfied: $[l_1] = [l_2]$ iff $l_1 \simeq l_2$ (where $\simeq$ is the equivalence relation of being parallel). But there is also what we may refer to as the #2 type of abstraction where the "direction of l" is an abstract object that captures what is common to parallel lines and *abstracts away from where they differ*.

The purpose of this paper is:

- to give a way to mathematically differentiate the #1 and #2 abstracts in a simple setting,
- to show that finite probability theory can be reformulated with the #2 abstracts replacing the #1 abstracts (i.e., the subsets as events), and then
- to show that the mathematical treatment of the #2 abstracts is essentially the same as is found in a rather different setting to describe superposition states in quantum mechanics–where the #2 abstracts-version of probability becomes quantum probability.

While this may add a little 'reality' to discussions of abstraction in the philosophy of mathematics, the main point is to build the bridge to QM and thus to better understand 'by analogy' the key superposition principle in QM.

2 Two Versions of Abstraction

One general form of an abstraction principle is given by Shapiro [19, p. 107]:

$$(\forall a)\,(\forall b)\,(\Sigma\,(a) = \Sigma\,(b) \equiv E\,(a, b)).$$

1. the #1 version of the abstraction operation takes equivalent entities $E\,(a, b)$ to the equivalence class $\Sigma\,(a) = [a] = [b] = \Sigma\,(b)$, and;
2. the #2 version of the abstraction operation takes all the equivalent entities a, b such that $E\,(a, b)$ to the abstract entity that is definite on what is common in the equivalence class but is indefinite on how they differ (e.g., on all the other properties that distinguish them).

In Frege's well-known example from the *Grundlagen* [10, pp. 110-111], an equivalence class of parallel lines is a #1 type of abstraction out of some delimited class of lines, while the act of abstracting away from the differences between parallel lines (i.e., going from equivalence to identity) yields the #2 abstraction of direction.

W. T. Tutte provides a good example of the attitude of a working mathematician.

> Pure graph theory is concerned with those properties of graphs that are invariant under isomorphism, for example the number of vertices, the number of loops, the number of links, and the number of vertices of a given valency. It is therefore natural for a graph theorist to identify two graphs that are isomorphic. For example, all link-graphs are isomorphic, and therefore he speaks of the 'link-graph' as though there were only one.

> Similarly one hears of 'the null graph', 'the vertex graph', and 'the graph of the cube'. When this language is used, it is really an isomorphism class (also called an *abstract graph*) that is under discussion. ([22, p. 6 (original emphasis)]; quoted in: [15, p. 390])

Tutte seems to be referring to both types of abstraction. For instance, a proof about a property of "the graph of the cube" is not a property of an isomorphism class of graphs but a property of the graphs in that class or of the "abstract graph" that abstracts away from the different instances in the isomorphism class. Often proofs that could be seen as using the #2 abstract graph are formulated using systematic ambiguity, i.e., assuming an arbitrary graph in the isomorphism class and then using only the properties common to all members of the class–which are precisely the properties in the #2 abstract graph.

Our purpose is to give clearly distinct models for these two types of abstracts, but first we might consider the two abstracts in a broader setting (without assuming an equivalence relation). This broader setting allows us to give a #2 abstract interpretation to "events" in finite probability theory–which, in turn, will facilitate the bridge over to QM.

Given *any* property $S(u)$ defined on the elements of U, two abstract objects can be defined as in Figure 1:

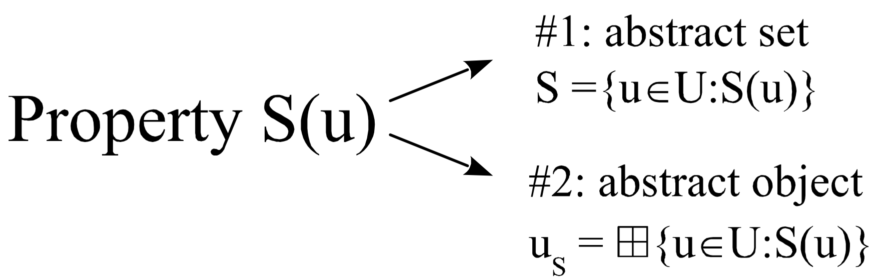

Figure 1: A property determines two types of abstract objects (the 'blob-sum' $\boxplus$ is defined below).

In the spirit of the working mathematician, the #1 abstract uses the naive comprehension scheme from what Paul Halmos called "naive set theory" [11] while the #2 abstract object u_S is '*the* S-entity' which is definite on the $S(u)$ property and indefinite on the differences between all the $u \in U$ such that $S(u)$.

We have a naming problem for these #2 abstracts like the problem of describing a glass as being half-full or half-empty. We could describe the #2 abstract u_S according to the properties that remain definite so it is a type of *paradigm* S-entity (the 'half-full' description), or we could describe the #2 abstract u_S as the *indefinite* S-entity that remains after all the properties that differentiate distinct S-entities are removed (the 'half-empty' description). For instance, in a logical context, the paradigm description might seem most appropriate while in the eventual application to quantum mechanics, it is the indefiniteness aspect of superposition states that is paramount.

3 An Example Starting with Properties

Consider three predicates (binary attributes) $P(x)$, $Q(x)$, and $R(x)$ which could distinguish at most $2^3 = 8$ definite-particular entities: $u_1, ..., u_8$ called *eigen-elements* and which can be presented in Table 1 like a truth table:

$P(x)$	$Q(x)$	$R(x)$	u
1	1	1	u_1
1	1	0	u_2
1	0	1	u_3
1	0	0	u_4
0	1	1	u_5
0	1	0	u_6
0	0	1	u_7
0	0	0	u_8

Table 1: Eight entities distinguished by 3 properties.

The general rule is if $f, g, h : U \to \mathbb{R}$ are numerical attributes with the number of distinct values as n_f, n_g, and n_h respectively, then those attributes could distinguish or classify $n_f \times n_g \times n_h$ distinct subsets of U. If the join of the inverse-image partitions is the discrete partition (all singletons), i.e.,$\{f^{-1}\} \vee \{g^{-1}\} \vee \{h^{-1}\} = \mathbf{1}_U$ as in Ellerman [6], then $\{f, g, h\}$ is a *complete set of attributes* since they can uniquely distinguish or classify the eigen-elements of U. Then we can distinguish the elements of U by their triple of values, i.e., $|f(u_j), g(u_j), h(u_j)\rangle$ uniquely determines $u_j \in U$.

In the example, any subset $S \subseteq U = \{u_1, ..., u_8\}$ is characterized by a property $S(x)$, the *disjunctive normal form property*, common to all and only the elements of S. If $S = \{u_1, u_4, u_7\}$, then the DNF property is:

$$S(x) = [P(x) \wedge Q(x) \wedge R(x)] \vee [P(x) \wedge \neg Q(x) \wedge \neg R(x)] \vee [\neg P(x) \wedge \neg Q(x) \wedge R(x)].$$

But what are the #1 and #2 abstract entities?

1. The #1 abstract entity is just the set

$$S = \{u_i \in U | S(u_i)\} = \{u_1, u_4, u_7\}$$

of all the distinct $S(x)$-entities; and

2. The #2 abstract entity is $S(x)$-entity symbolized

$$u_S = u_1 \boxplus u_4 \boxplus u_7 = \boxplus\{u_i \in U | S(u_i)\}$$

The 'superposition' or 'blob-sum' of u_1, u_4, and u_7.

that is *definite on the DNF property* $S(x)$ but indefinite on what distinguishes the different $S(x)$-entities.

4 Some Philosophical Concerns

It is best to think of S as the set of *definite particular* $S(x)$-entities in some universe U, while u_S is the *indefinite paradigm-universal* $S(x)$-entity that is the 'superposition' or blob-sum $u_S = \boxplus\{u_i \in U | S(u_i)\}$. In general, the #2 abstract u_S is "One *over* the Many." Only when $S = \{u_j\}$ is a singleton does the definite description '*the* S-entity' refer to an element of U, i.e., $u_{\{u_j\}} = u_j$. As in postulating #1 abstracts (sets) for properties $S(x)$, a working mathematician has to be careful about the properties allowed for #2 abstracts–as in the last section where only combinations of $P(x)$, $Q(x)$, and $R(x)$ were allowed.

Making the "One" $u_S = \boxplus\{u_i \in U | S(u_i)\}$ "*over* the Many," i.e., more abstract than the $u_i \in U$ (for $|S| > 1$) avoids the paradoxes just as the iterative notion of set does in ordinary set theory, i.e., for #1 type of abstractions. Otherwise, if we ignore the given set U, then we can recreate Russell's Paradox for #2 abstracts. Let $R(u_S) \equiv \neg S(u_S)$ so:

$$u_R = \boxplus \{u_S | \neg S(u_S)\} \text{ and thus } R(u_R) \text{ implies } \neg R(u_R), \text{ and } \neg R(u_R) \text{ implies } R(u_R).$$

But if we define $u_R = \boxplus \{u_S \in U | \neg S(u_S)\}$, then assuming $u_R \in U$ just leads to the contradiction so $u_R \notin U$.

The paradigm-universal u_S should not be thought of as universal 'S-ness'. Intuitively, if $S(x)$ is having the color white, then u_{white} = '*the* white thing', not 'whiteness'. According the William and Martha Kneale, this distinction (or confusion) goes back to Plato:

> But Plato also used language which suggests not only that the Forms exist separately ($\chi\omega\rho\iota\sigma\tau\alpha$) from all the particulars, but also that each Form is a peculiarly accurate or good particular of its own kind, i.e., the standard particular of the kind in question or the model ($\pi\alpha\rho\alpha\delta\varepsilon\iota\gamma\mu\alpha$) to which other particulars approximate. [13, p. 19]

Some have considered interpreting the Form as *paradeigma* as an error.

> For general characters are not characterized by themselves: humanity is not human. The mistake is encouraged by the fact that in Greek the same phrase may signify both the concrete and the abstract, e.g. $\lambda\varepsilon\upsilon\kappa o\nu$ (literally "the white") both "the white thing" and "whiteness", so that it is doubtful whether $\alpha\upsilon\tau o\ \tau o\ \lambda\varepsilon\upsilon\kappa o\nu$ (literally "the white itself") means "the superlatively white thing" or "whiteness in abstraction". [13, pp. 19-20]

5 Relations Between #1 and #2 Universals

In the version of finite probability theory developed below, the #2 paradigm-universals u_S will replace the #1 universals or events $S \subseteq U$. The set of events is the power-set $\wp(U)$, and it is replaced by the set of paradigm entities $\{u_S | S \in \wp(U)\}$. Hence we first show how to translate between the two versions of universals.

For properties $S()$ defined on U, there is a 1-1 correspondence between the #1 and #2 universals:

$$\cup \{\{u_j\} | u_j \in U \& S(u_j)\} = S \longleftrightarrow u_S = \boxplus \left\{u_{\{u_j\}} | u_j \in U \& S(u_j)\right\}.$$

If $T()$ another property defined on U implies $S()$ in the sense that $(\forall u \in U)[T(u) \Rightarrow S(u)]$, then in terms of #1 abstracts, this is the familiar $T \subseteq S$.

But what is the #2 universals equivalent of $T \subseteq S$? Intuitively u_S is '*the* S-thing' that is definite on the S-property but is otherwise indefinite on the differences between the members of S. Those differences have been abstracted away from, blurred or 'blobbed' out, or rendered indefinite. If we make more properties definite, then in terms of subsets, that will in general cut down to a subset $T \subseteq S$, so u_T would be a more definite version of u_S. This "process" of changing from u_S to a more definite universal u_T, i.e., $u_S \rightsquigarrow u_T$ for $T \subseteq S$, might be called *projection* or *sharpening* (as in making a camera focus sharper or more definite) and symbolized:

$$u_S \rhd u_T \text{ (or } u_T \lhd u_S)$$
$$u_S \text{ can be "sharpened" to } u_T \text{ by adding some definiteness.}$$

These relations between #1 and #2 abstracts are summarized in Table 2.

$S()$ defined on U	#1 abstraction	#2 abstraction
Universals for $S()$	$S = \cup \{\{u_j\} \| u_j \in U \& S(u_j)\}$	$u_S = \boxplus \left\{u_{\{u_j\}} \| u_j \in U \& S(u_j)\right\}$
$T()$ implies $S()$	$T \subseteq S$	$u_S \rhd u_T$

Table 2: Equivalents between #1 and #2 universals

In the language of Plato, the projection relation $\lhd$ is the relation of "participation" ($\mu\varepsilon\theta\varepsilon\xi\iota\varsigma$ or *methexis*) or entailment between universals. As Plato would say, '*the* T-thing' participates in or 'brings-on' ($\epsilon\pi\iota\phi\epsilon\pi\epsilon\iota$ or *epipherei* as in Vlastos [24, p. 102]) '*the* S-thing,' as in '*the* rocking chair' brings on '*the* chair,' i.e., $u_T \lhd u_S$, since '*the* chair' can be sharpened to '*the* rocking chair.'[1]

[1] These non-mathematical everyday examples are used for the purpose of illustration and, perhaps, amusement.

Thus we have described two types of abstract objects:

1. Axiomatic set theory is the formal theory of #1 abstract objects, the sets S, where taking $\in$ as the participation relation, sets are *never* self-participating, i.e., $S \notin S$;

2. There could be a second theory about the #2 abstract entities (described here "naively"), the paradigms u_S, which are *always* self-participating, i.e., $u_S \lhd u_S$ (i.e., u_S is the null-sharpening of u_S).

Like the #1 abstracts S, the #2 abstract entities u_S, the *paradigm-universals*, are also used in mathematics.

6 Examples of Abstract Paradigms in Mathematics

There is an equivalence relation $A \simeq B$ between topological spaces which is realized by a continuous map $f : A \to B$ such that there is an inverse $g : B \to A$ so the $fg : B \to B$ is homotopic to 1_B (i.e., can be continuously deformed in 1_B) and gf is homotopic to 1_A. According to the 'classical' homotopy theorist, Hans-Joachim Baues, "Homotopy types are the equivalence classes of spaces" [2] under this equivalence relation. That is the #1 type of abstraction.

But the interpretation offered in homotopy type theory (HoTT) is expanding identity to "coincide with the (unchanged) notion of equivalence" in the words of the Univalent Foundations Program [23, p. 5] so it would refer to the #2 homotopy type, i.e., '*the* homotopy type' that captures the mathematical properties shared by all spaces in an equivalence class of homotopic spaces (wiping out the differences). Expanding identity to coincide with equivalence is another way to describe the #2 abstracting from the class S of equivalent entities to the abstract paradigm-universal entity u_S which is not the same as the particular entities u in the equivalence class S.

For instance, '*the* homotopy type' is not one of the classical topological spaces (with points etc.) in the #1 equivalence class of homotopic spaces–just as Frege's #2 abstraction of direction is not among the lines in the equivalence class of parallel lines with the *same* direction.

> While classical homotopy theory is analytic (spaces and paths are made of points), homotopy type theory is synthetic: points, paths, and paths between paths are basic, indivisible, primitive notions. [23, p. 59]

Homotopy type theory systematically develops a theory of the #2 type of abstractions that grows out of homotopy theory and type theory into a new foundational theory.[2]

> From the logical point of view, however, it is a radically new idea: it says that isomorphic things can be identified! Mathematicians are of course used to identifying isomorphic structures in practice, but they generally do so by "abuse of notation", or some other informal device, knowing that the objects involved are not "really" identical. But in this new foundational scheme, such structures can be formally identified, in the logical sense that every property or construction involving one also applies to the other. [23, p. 5]

In our terminology, "isomorphic things can be identified" is 'blobbing together' of all the elements in an isomorphism class to create a single #2 abstract that is definite on what is common to all the isomorphs but is indefinite on where they differ.

[2] Here we only develop #2 abstracts informally in the same sense that the #1 abstracts, sets, are used in naive set theory [11]. But homotopy type theory can be seen as one way to have a formal theory of these #2 abstractions in mathematics–although the interpretation of HoTT is subect to controversy, e.g., Ladyman and Presnell's [14]. Our naive and speculative development of #2 abstracts is not intended to illuminate the already-rigorously developed HoTT, but to build a conceptual bridge to QM.

Consider the homotopy example of '*the* path going once (clockwise) around the hole' in an annulus A (disk with one hole as in Figure 2), i.e., the abstract entity 1 in the fundamental group $\pi_0(A)$ of the annulus: $1 \in \pi_0(A) \cong \mathbb{Z}$:

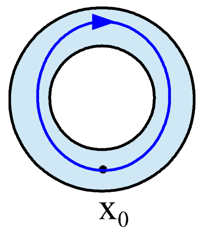

Figure 2: '*the* path going once (clockwise) around the hole'

Note that '*the* path going once (clockwise) around the hole' has the paradigmatic property of "going once (clockwise) around the hole" but is *not* one of the particular (coordinatized) paths that constitute the equivalence class of coordinatized once-around paths deformable into one another.

In a similar manner, we can view other common #2 abstractions such as: '*the* cardinal number 5' that captures what is common to the isomorphism class of all five-element sets; '*the* integer 1 mod (n)' that captures what is common within the equivalence class $\{..., -2n+1, -n+1, 1, n+1, 2n+1, ...\}$ of integers; '*the* circle' or '*the* equilateral triangle'–and so forth.

Category theory helped to motivate homotopy type theory for good reason. Category theory has no notion of identity between objects, only isomorphism as 'equivalence' between objects. Therefore category theory can be seen as a theory of *abstract* #2 objects (i.e., the #2 abstract of an isomorphism class), e.g., abstract sets, groups, spaces, etc.

Our purpose is to model the theory of paradigm-universals u_S and their projections or sharpenings u_T–that is analogous to working with sets and subsets, e.g., in a Boolean algebra of subsets. That is all we will need to show that probability theory can be developed using paradigm entities u_S instead of subset-events S, and then finally to cross the bridge to quantum mechanics.

7 The Connection to Interpreting Symmetry Operations

In the usual case of abstraction where S is an equivalence or isomorphism class, the #2 universal u_S by definition abstracts away for the differences between the elements in the equivalence class. Hence if we consider any operation that takes one element u of an equivalence class $[u]$ to another element u' in the same class, then the induced operation on the #1 abstracts, $[u] \rightsquigarrow [u']$, is the identity, and the *same* holds for the #2 abstract u_S since the two abstracts represent two different ways to get abstracts that in different ways disregard the differences between the elements in the equivalence class.

This can be visually illustrated in a simple example of the symmetry operation (defining an equivalence relation) of reflection on the aA-axis for a fully definite isosceles triangles as in Figure 3.

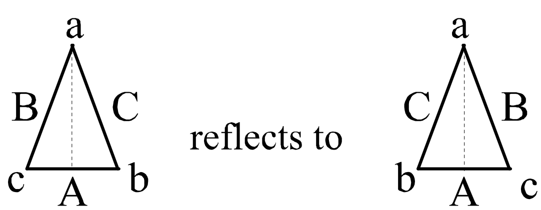

Figure 3: Reflection on vertical axis symmetry operation.

Thus the equivalence class of reflective-symmetric figures in the #1 or classical interpretation is the set in Figure 4.

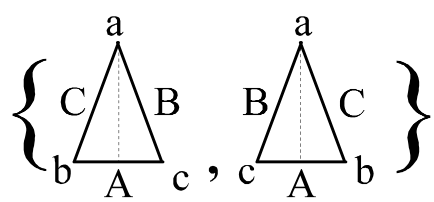

Figure 4: The #1 abstraction of equivalence class.

The set remains invariant under reflection applied to its elements, which is another way to say that the induced operation *on the equivalence classes* (or orbits) is the identity.

Under the #2 indefiniteness-abstraction interpretation, the equivalence abstracts to the figure that is definite as to what is the same, and indefinite as to what is different between the definite figures in the equivalence class:

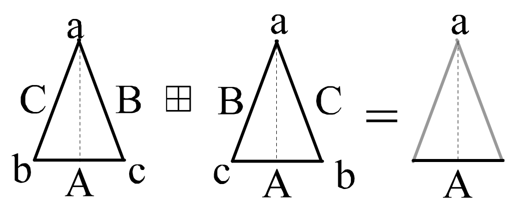

Figure 5: The #2 abstraction of an indefinite entity.

And the symmetry operation induced on the indefinite figure is also the *identity* as illustrated in Figure 5. As noted in the discussion of homotopy type theory, the movement from one space to a homotopic space leaves the "homotopy type" *the same* regardless of whether we think of the homotopy type as an equivalence class or as the #2 type of abstract considered in homotopy type theory.

The notion of "indiscernibility" is deliberately avoided here since it can be something of a philosophical weasel-word. For instance, in the philosophical discussions about abstraction, identity, and *ante rem* structuralism (e.g., John Burgess [3]; Jukka Keränen [12]; Fraser MacBride and the authors in [16]; Roy Cook and the authors in [5]; Hannes Leibgeb and James Ladyman [15]; and Stewart Shapiro [18], [19]), "indiscernibility" is a function of what constants, predicates, and relations we allow in the descriptive language. In one standard example, $+1$ and -1 are structurally indiscernible in the additive group $(\mathbb{Z}, +, 0)$ but not if we allow the constant 2 in the language since $x + x = 2$ distinguishes $+1$ and -1. Authors discuss automorphisms distinguished from the identity but on indistinguishable elements, so clearly there are different notions of "indiscernibility" or "indistinguishability" in play. An automorphism $\alpha : U \to U$ can only be distinguished from the identity 1_U if there are at least two "indistinguishable" elements u and $\alpha(u)$ that can be distinguished. Since we are later going to relate the #2 abstract entities to the indefinite states of quantum mechanics, the differentiation of classical statistics from quantum statistics should not be based on our language-dependent notions of classical and quantum "indiscernibility" or "indistinguishability."

One example is the derivation of the Maxwell-Boltzmann distribution and the Bose-Einstein distribution as in Feller [9, pp. 20-1] or Ellerman [7]. This treatment is illuminated by the classical and quantum version of a symmetry operation. Suppose we have two particles of the same type which are classically indistinguishable so, following Weyl, we artificially distinguish them using Mike and Ike labels. If each of the two particles could be in states A, B, or C, then the set of possible states is the set of nine ordered pairs $\{A,B,C\} \times \{A,B,C\}$. Applying the symmetry operation of permuting Mike and Ike, we have six equivalence classes (orbits) as in Table 3.

Equivalence classes under permutation	M-B
$\{(A,B),(B,A)\}$	2/9
$\{(A,C),(C,A)\}$	2/9
$\{(B,C),(C,B)\}$	2/9
$\{(A,A)\}$	1/9
$\{(B,B)\}$	1/9
$\{(C,C)\}$	1/9

Table 3: Maxwell-Boltzmann distribution.

The symmetry operation on the equivalence classes is the identity, but in Nature the primitive data are, as it were, the ordered pairs (the possible states), not the equivalence classes. When we assign the equal probabilities of $\frac{1}{9}$ to each ordered pair (i.e., to each distinct state), that results in the *Maxwell-Boltzmann distribution* on the equivalence classes. Nature counts states; we classically measure equivalence classes and find the M-B distribution.

But in the quantum case, the operation of going to the #2 abstract $u_{\{(A,B),(B,A)\}}$ seems to be physically realized in an indefinite superposition state, i.e., the analogy: $u_{\{(A,B),(B,A)\}} \approx \frac{1}{\sqrt{2}}[|A,B\rangle + |B,A\rangle]$, where the symmetry operation is the *identity*. Since there are *then* only six states, we assign the equal probabilities of $\frac{1}{6}$ to each state and obtain the *Bose-Einstein distribution* in Table 4. Nature again counts states, but the superposition states (seen as physically realizing a type of #2 abstract from the equivalence classes) reduces the number of states to six.

Six indefinite states	B-E
$u_{\{(A,B),(B,A)\}} \approx \frac{1}{\sqrt{2}}[|A,B\rangle + |B,A\rangle]$	1/6
$u_{\{(A,C),(C,A)\}} \approx \frac{1}{\sqrt{2}}[|A,C\rangle + |C,A\rangle]$	1/6
$u_{\{(B,C),(C,B)\}} \approx \frac{1}{\sqrt{2}}[|B,C\rangle + |C,B\rangle]$	1/6
$u_{\{(A,A)\}} \approx |A,A\rangle$	1/6
$u_{\{(B,B)\}} \approx |B,B\rangle$	1/6
$u_{\{(C,C)\}} \approx |C,C\rangle$	1/6

Table 4: Bose-Einstein distribution.

8 Modelling #1 and #2 Abstracts to get Paradigms Probability Theory

But it will surely be asked:

> What is this crazy talk and loose analogy between forming an indefinite abstract in mathematics and a superposition state in QM?

It is a fine question, and surely one way to approach the question is to give 'clear and distinct' mathematical models of the two abstracts in a simple illustrative setting. We distinguish the #1 and #2 interpretations for a finite U as in Figure 6.

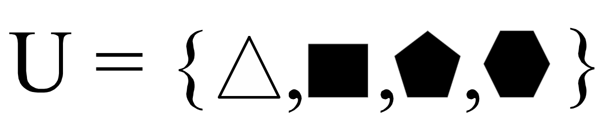

Figure 6: Universe U of figures

The polygons in Figure 6 can be characterized using two attributes, the number n of equal sides and being solid s or hollow h. Hence the universe U has the elements $U = \{u_1, u_2, u_3, u_4\} = \{3h, 4s, 5s, 6s\}$. Ordinarily the subset of *solid* figures $S = \{4s, 5s, 6s\} \subseteq \{3h, 4s, 5s, 6s\} = U$ would be represented by a one-dimensional column vector $|S\rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{matrix} 3h \\ 4s \\ 5s \\ 6s \end{matrix}$ (with the given ordering). But by moving up one dimension to a *two*-dimensional matrix, we can *represent* or *mathematically model* the *two* #1 and #2 versions of S as two types of incidence matrices. For $U = \{u_1, ..., u_n\}$, the *incidence matrix* $\mathrm{In}(R)$ of a binary relation $R \subseteq U \times U$ is the $n \times n$ matrix with $(\mathrm{In}(R))_{jk} = 1$ if $(u_j, u_k) \in R$ and 0 otherwise.

1. The #1 (classical) representation of S (i.e., set of S-things or set of solid figures) is the diagonal matrix $\mathrm{In}(\Delta S)$ that lays the column vector $|S\rangle$ along the diagonal: $\mathrm{In}(\Delta S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$ representation of set S of distinct S-entities. $\mathrm{In}(\Delta S)$ is the incidence matrix of the diagonal relation $\Delta S \subseteq U \times U$ whose entries are the values of the characteristic function $\chi_{\Delta S}$ on $U \times U$.

2. The #2 (quantum-like) representation of S (i.e., *the* S-thing) is the matrix $\mathrm{In}(S \times S)$ whose entries are the values of the characteristic function $\chi_{S \times S}$ on $U \times U$. Where $()^t$ signifies the transpose operation, this $n \times n$ incidence matrix can also be obtained as the product of the $n \times 1$ column vector $|S\rangle$ times the $1 \times n$ row vector $(|S\rangle)^t$: $\mathrm{In}(S \times S) = |S\rangle (|S\rangle)^t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} =$ representation of one indistinct S-thing, '*the* solid figure' $u_S = 4s \boxplus 5s \boxplus 6s$.

Recall that for (and only for) singletons $S = \{u_j\}$, the #2 'abstract' is just u_j, and thus they have the same representation $\mathrm{In}(\Delta S) = \mathrm{In}(S \times S)$ as expected, but for $|S| > 1$, $\mathrm{In}(\Delta S) \neq \mathrm{In}(S \times S)$.

The two representations differ only in the off-diagonal entries. Think of the off-diagonal $\mathrm{In}(S \times S)_{j,k} = 1$'s as equating, cohering, blurring out, or 'blobbing' out the differences between u_j and u_k which have the common $S() =$ 'being a solid figure' property:

$$\mathrm{In}(S \times S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \text{ says}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4s \overset{S}{\sim} 4s & 4s \overset{S}{\sim} 5s & 4s \overset{S}{\sim} 6s \\ 0 & 5s \overset{S}{\sim} 4s & 5s \overset{S}{\sim} 5s & 5s \overset{S}{\sim} 6s \\ 0 & 6s \overset{S}{\sim} 4s & 6s \overset{S}{\sim} 5s & 6s \overset{S}{\sim} 6s \end{bmatrix}.$$

Intuitively, the differences in the number of sides of the solid figures have been blurred out or rendered indefinite, so the only definite attribute of the paradigm entity is *the* solid-figure.

Since the #2 abstract paradigm entities are represened by a certain type of incidence matrix, we can mathematically represent the *blob-sum* #2 operation on entities (used above only intuitively): $u_S = \boxplus \{u_i \in U | S(u_i)\}$ is represented by the *blob-sum* $\boxplus$ of the corresponding incidence matrices:

$$\boxplus_{u_i \in S} \mathrm{In}(\{u_i\} \times \{u_i\}) =_{df} \mathrm{In}(S \times S)$$

where the incidence-matrix *blob-sum* $\boxplus$ is defined for $S_1, S_2 \subseteq U$ with $S = S_1 \cup S_2$:

$$\begin{gathered}
\mathrm{In}\,(S \times S) = \mathrm{In}\,(S_1 \times S_1) \boxplus \mathrm{In}\,(S_2 \times S_2) = \mathrm{In}\,((S_1 \cup S_2) \times (S_1 \cup S_2)) \\
= \mathrm{In}\,((S_1 \times S_1) \cup (S_2 \times S_2) \cup (S_1 \times S_2) \cup (S_2 \times S_1)) \\
= \mathrm{In}\,(S_1 \times S_1) \vee \mathrm{In}\,(S_2 \times S_2) \vee \mathrm{In}\,(S_1 \times S_2) \vee \mathrm{In}\,(S_2 \times S_1). \\
\mathrm{In}\,(S \times S) = \mathrm{In}\,(S_1 \times S_1) \vee \mathrm{In}\,(S_2 \times S_2) \vee \text{ blobbing cross-terms.}^3
\end{gathered}$$

For $S = \{u_2, u_4\}$, the blob-sum $u_S = u_2 \boxplus u_4$ is represented by:

$$\mathrm{In}\,(\{u_2\} \times \{u_2\}) \boxplus \mathrm{In}\,(\{u_4\} \times \{u_4\}) = \mathrm{In}\,(S \times S)$$

where the blob-sum operation $\boxplus$ means 'blobbing-out' the distinctions between entities in S (given by the cross-terms in $\{u_2, u_4\} \times \{u_2, u_4\}$):

$$\begin{gathered}
\mathrm{In}\,(S \times S) = \mathrm{In}\,(\{u_2\} \times \{u_2\}) \boxplus \mathrm{In}\,(\{u_4\} \times \{u_4\}) \\
= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boxplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
= \mathrm{In}\,(\{u_2, u_4\} \times \{u_2, u_4\}) \\
= \mathrm{In}\,(\{u_2\} \times \{u_2\}) \vee \mathrm{In}\,(\{u_4\} \times \{u_4\}) \vee \mathrm{In}\,(\{u_2\} \times \{u_4\}) \vee \mathrm{In}\,(\{u_4\} \times \{u_2\}) \\
= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.
\end{gathered}$$

Due to the development of Boolean subset logic and set theory, we are perfectly comfortable with considering the #1 abstractions of sets S of even concrete ur-elements like the *set* of chairs in a room. The representatives $\mathrm{In}\,(\Delta S)$ trivially form a BA isomorphic to the BA of subsets $S \in \wp(U)$.

To better understand abstraction in mathematics, the paradigm-version of probability theory (defined below), and superposition states in QM, we should become as comfortable with paradigms u_S as with subsets S. The paradigms u_S for $S \in \wp(U)$ form a Boolean algebra isomorphic to $\wp(U)$ under the mapping: for any Boolean binary operation $S \# T$ for $S, T \in \wp(U)$, $u_S \# u_T$ is the paradigm represented by $\mathrm{In}\,((S \# T) \times (S \# T))$.

- The *union or join of paradigms* is the blob-sum $u_{S \cup T} = u_S \boxplus u_T$ which is the #2 abstract or paradigm represented by $\mathrm{In}\,(S \times S) \boxplus \mathrm{In}\,(T \times T) = \mathrm{In}\,((S \cup T) \times (S \cup T))$ (note as expected, for $T \subseteq S$, $u_S \boxplus u_T = u_S$);

- The *intersection or meet of paradigms* $u_S \wedge u_T = u_{S \cap T}$ is represented by $\mathrm{In}\,(S \times S) \wedge \mathrm{In}\,(T \times T) = \mathrm{In}\,((S \cap T) \times (S \cap T))$ (the meet $\wedge$ of incidence matrices is just the entry-wise conjunction of the $0,1$-entries) where, as expected, for $T \subseteq S$, $u_S \wedge u_T = u_T$;

- The *negation of a paradigm* $\neg u_S = u_{S^c}$ is represented by $\mathrm{In}\,(S^c \times S^c) = \boxplus \{\mathrm{In}\,(\{u\} \times \{u\}) \,|\, u \notin S\}$ (note as expected, $u_S \boxplus u_{S^c} = u_{S \cup S^c} = u_U$).

The top u_U and bottom $u_\emptyset$ of the BA are represented by the incidence matrices of all ones or all zeros respectively, and the partial order on the blobbed-out incidence matrices $\mathrm{In}\,(S \times S)$ is that induced by set inclusion [i.e., the entry-wise partial order $0 \leq 1$ on incidence matrices of the form $\mathrm{In}\,(S \times S)$]. If $T \subseteq S$, then $u_T \triangleleft u_S$, so moving down in the BA of paradigms represents 'sharpening' or rendering-more-definite just as a conditional probability $\Pr(T|S)$ is always for some event T (or

[3] The disjunction of incidence matrices is the usual entry-wise disjunction: $1 \vee 1 = 1 \vee 0 = 0 \vee 1 = 1$ and $0 \vee 0 = 0$, and similarly for conjunction.

$T \cap S$) below the conditioning event S in the partial order of events. The atomic elements $u_{\{u_i\}}$ (corresponding to the singletons $\{u_i\}$) are the sharpest or most definite elements. When the events as subsets S of the sample space U, are replaced by the #2 abstracts u_S, then this Boolean algebra structure on the set of paradigms u_S in their $\text{In}(S \times S)$ representation for $S \subseteq U$ replaces the usual BA of events S. Figure 7 illustrates the two BAs for $U = \{a, b, c\}$.

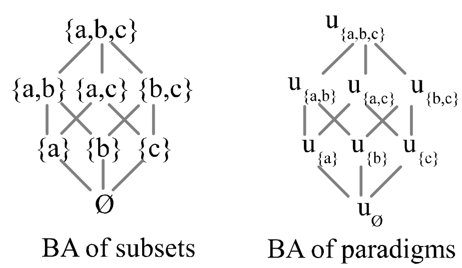

Figure 7: The Boolean algebras of events and paradigms for $U = \{a, b, c\}$.

9 The Projection Operation: Making an indefinite entity more definite

In the example of four figures, suppose we classify or partition *all* the elements of U according to an attribute such as the parity of the number of sides, where a *partition* is a set of disjoint subsets (blocks) of U whose union is all of U. Let π be the partition of U with two *blocks* $O = \{Odd\} = \{3h, 5s\}$ and $E = \{Even\} = \{4s, 6s\}$ according to the parity of the number of sides.

The equivalence relation defined by π is referred to by Ellerman [6] as the set of *indistinctions*, $\text{indit}(\pi) = (O \times O) \cup (E \times E)$, and the incidence matrix $\text{In}(\text{indit}(\pi))$ is formed by the usual disjunction of corresponding matrix entries:

$$\text{In}(O \times O) \vee \text{In}(E \times E) = \text{In}(\text{indit}(\pi))$$
$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The #1 (classical) operation of intersecting the set of odd-sided figures with the set of solid figures to give the set of odd-sided solid figures is represented as the conjunction:

$$\text{In}(\Delta O) \wedge \text{In}(\Delta S) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The #2 (quantum-like) operation of 'sharpening' or 'rendering more definite' '*the* solid figure' $u_S = u_{\{u_2,u_3,u_4\}}$ to '*the* odd-sided solid figure' $u_{\{u_s\}} = u_{\{5s\}} = 5s$, so $u_{\{5s\}} \lhd u_S$ (suggested reading: $u_{\{5s\}}$ is a projection or sharpening of u_S) is represented as:

$$\text{In}(O \times O) \wedge \text{In}(S \times S) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = u_{\{5s\}}.$$

But there is a better way to represent 'sharpening' using matrix multiplication instead of just the logical operation $\wedge$ on matrices, and it foreshadows and illuminates the measurement operation in QM. The matrix $\mathrm{In}(\Delta E) = P_E$ is a projection matrix, i.e., the diagonal matrix with diagonal entries $\chi_E(u_i)$ so $P_E|S\rangle = |E \cap S\rangle$. Then the result of the projection-sharpening can be represented as:

$$\begin{aligned}|E \cap S\rangle (|E \cap S\rangle)^t &= P_E|S\rangle (P_E|S\rangle)^t = P_E|S\rangle (|S\rangle)^t P_E \\ &= P_E \mathrm{In}(S \times S) P_E = \mathrm{In}(E \times E) \wedge \mathrm{In}(S \times S).\end{aligned}$$

Thus sharpening *the* solid-figure $u_{\{4s,5s,6s\}}$ by the even number-of-sides attribute to obtain $u_{\{4s,6s\}}$ is represented by pre- and post-multiplying the incidence matrix $\mathrm{In}(S \times S)$ by the projection P_E for evenness parity. Under the #2 interpretation, the parity-sharpening, parity-classifying, parity-differentiation, or *parity-measurement* of '*the* solid figure' by *both* the odd and even parities is represented as:

$$\begin{aligned}\mathrm{In}(\mathrm{indit}(\pi)) \wedge \mathrm{In}(S \times S) &= P_O \mathrm{In}(S \times S) P_O + P_E \mathrm{In}(S \times S) P_E \\ &= \begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix} \wedge \begin{bmatrix}0&0&0&0\\0&1&1&1\\0&1&1&1\\0&1&1&1\end{bmatrix} \\ &= \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&1&0&1\\0&0&0&0\\0&1&0&1\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&1&0&1\\0&0&1&0\\0&1&0&1\end{bmatrix}.\end{aligned}$$[4]

The result is the mixture or sum (*not* blob-sum) of incidence matrices for '*the* even-sided solid figure' $u_{\{u_2,u_4\}} = u_{\{4s,6s\}}$ and '*the* odd-sided solid figure' $u_{\{u_3\}} = u_{\{5s\}} = 5s$. The important thing to notice is the action on the off-diagonal elements where the action $1 \rightsquigarrow 0$ in the j,k-entry means that a *distinction* between u_j and u_k has been created; u_j and u_k have been deblobbed, decohered, distinguished, or differentiated–in this case by parity in the number of sides:

$$\begin{aligned}&\mathrm{In}(S \times S) \rightsquigarrow \mathrm{In}(\mathrm{indit}(\pi)) \wedge \mathrm{In}(S \times S) \\ &= P_O \mathrm{In}(S \times S) P_O + P_E \mathrm{In}(S \times S) P_E \\ &= \begin{bmatrix}0&0&0&0\\0&1&1\overset{decohered}{\rightsquigarrow}0&1\\0&1\overset{decohered}{\rightsquigarrow}0&1&1\overset{decohered}{\rightsquigarrow}0\\0&1&1\overset{decohered}{\rightsquigarrow}0&1\end{bmatrix}\end{aligned}$$

Differentiating by parity.

In terms of the logical notion of information-as-distinctions as in Ellerman [8], the non-zero off-diagonal terms that are zeroed in the classification or measurement process give the increase in logical entropy.

We could also classify the figures as to having 4 or fewer sides (few sides) or more (many sides) so that partition is $\sigma = \{\{u_1,u_2\},\{u_3,u_4\}\} = \{\{3h,4s\},\{5s,6s\}\}$ which is represented by:

$$\mathrm{In}(\mathrm{indit}(\sigma)) = \begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix} \text{ and}$$

[4]This classifying or measuring operation using the pre- and post-multiplication by projection matrices foreshadows the Lüders mixture representation of projective measurement in QM (see below).

$$\begin{aligned}&\mathrm{In}\left(\mathrm{indit}\left(\sigma\right)\right)\wedge\left(\mathrm{In}\left(\mathrm{indit}\left(\pi\right)\right)\wedge\mathrm{In}\left(S\times S\right)\right)\\&=P_{few}\left(\mathrm{In}\left(\mathrm{indit}\left(\pi\right)\right)\wedge\mathrm{In}\left(S\times S\right)\right)P_{few}+P_{many}\left(\mathrm{In}\left(\mathrm{indit}\left(\pi\right)\right)\wedge\mathrm{In}\left(S\times S\right)\right)P_{many}\\&=\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\mathrm{In}\left(\Delta S\right).\end{aligned}$$

Thus the parity and the few-or-many-sides properties suffice to classify the solid figures uniquely and thus to yield the representation $\mathrm{In}\left(\Delta S\right)$ of the distinct elements of $S=\{u_2,u_3,u_4\}=\{4s,5s,6s\}$. Thus making all the distinctions (i.e., decohering the entities that cohered together in u_S) takes $\mathrm{In}\left(S\times S\right)\rightsquigarrow\mathrm{In}\left(\Delta S\right)$.

In QM jargon, the parity and few-or-many-sides attributes constitute a "complete set of commuting operators" (CSCO) so that measurement of the 'pure,' blobbed-out, superposition figure, '*the* solid figure,' by those observables will sharpen '*the* solid figure,' to the 'mixture' of the three separate solid eigen-figures:

- '*the* few- and even-sided solid figure' (the square $u_2=4s$),
- '*the* many- and odd-sided solid figure' (the pentagon $u_3=5s$), and
- '*the* many- and even-sided solid figure' (the hexagon $u_4=6s$).

10 From Incidence to Density Matrices

To move from Boolean logic to probability theory for paradigms, we move from incidence matrices to density matrices. The incidence matrices $\mathrm{In}\left(\Delta S\right)$ and $\mathrm{In}\left(S\times S\right)$ can be turned into *density matrices* by dividing through by their trace (sum of diagonal elements):

$$\rho\left(\Delta S\right)=\frac{1}{\mathrm{tr}[\mathrm{In}(\Delta S)]}\,\mathrm{In}\left(\Delta S\right)\text{ and }\rho\left(S\right)=\frac{1}{\mathrm{tr}[\mathrm{In}(S\times S)]}\,\mathrm{In}\left(S\times S\right).$$

In terms of probabilities, this means treating the outcomes in S as being equiprobable with probability $\frac{1}{|S|}$. But now we have the #1 and #2 interpretations of the sample space for finite discrete probability theory.

1. The #1 interpretation, represented by $\rho\left(\Delta U\right)$, is the classical version with U as the sample space of outcomes. For instance, the 6×6 diagonal matrix with diagonal entries $\frac{1}{6}$ is "the statistical mixture describing the state of a classical dice [die] before the outcome of the throw" [1, p. 176];
2. The #2 interpretation replaces the "sample space" with the one indefinite '*the* sample outcome' u_U represented by $\rho\left(U\right)$ (a 6×6 matrix with the $\frac{1}{6}$ diagonal entries 'blobbed out' to fill the whole matrix with $\frac{1}{6}$ entries) and, in a trial, the indefinite outcome u_U 'sharpens to' or becomes a definite outcome $u_{\{u_i\}}=u_i\in U$ with probability $\frac{1}{|U|}$.

Let $f:U\rightarrow\mathbb{R}$ be a real-valued random variable with distinct values ϕ_i for $i=1,...,m$ and let $\pi=\{B_i\}_{i=1,...,m}$ where $B_i=f^{-1}\left(\phi_i\right)$, be the partition of U according to the f-values as in Ellerman [7]. The classification or differentiation of $\rho\left(S\right)$ according to the different values is: $\mathrm{In}\left(\mathrm{indit}\left(\pi\right)\right)\wedge\rho\left(S\right)$ which distinguishes the elements of S that have different f-values. If P_{B_i} is the diagonal (projection) matrix with diagonal elements $\left(P_{B_i}\right)_{jj}=\chi_{B_i}\left(u_j\right)$, then the classified, differentiated, or measured density matrix is also obtained by the *Lüders mixture operation* of pre- and post-multiplying $\rho\left(S\right)$ by the projection matrices P_{B_i} [1, p. 279]:

$$\mathrm{In}\left(\mathrm{indit}\left(\pi\right)\right)\wedge\rho\left(S\right)=\sum_{i=1}^{m}P_{B_i}\rho\left(S\right)P_{B_i},$$

and the probability of a trial returning ϕ_i is:

$$\Pr(\phi_i|S) = \operatorname{tr}[P_{B_i}\rho(S)].$$

There are two interpretations of that probability corresponding to the #1 or #2 abstracts:

1. It is the probability that given the #1 abstract, i.e., the event S, a trial leads to the #1 abstract, the event $B_i \cap S$, occurring, or
2. It is the probability that given the #2 abstract, i.e., the entity u_S, a trial leads to (or sharpens to) the #2 abstract, the entity $u_{B_i \cap S}$.

For instance, in the previous example, where $f : U \to \mathbb{R}$ gives the parity partition π with the two values ϕ_{odd} and ϕ_{even}, then:

$$P_{even}\rho(S)P_{even} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

so $\operatorname{tr}[P_{even}\rho(S)] = \frac{2}{3}$ which, under the #2 (quantum-like) interpretation, is the conditional probability that a trial sharpens '*the* solid figure' to '*the* even-sided solid figure'. And under the #1 (standard) interpretation, $\Pr(\phi_{even}|S) = \operatorname{tr}[P_{even}\rho(\Delta S)] = \frac{2}{3}$ is the probability of a trial yielding *an* even-sided solid figure starting with the *set* of equiprobable solid figures represented by $\rho(\Delta S)$. Thus we have two different interpretations of finite probability theory, the conventional one using the #1 abstracts or events S and the new paradigm interpretation using #2 abstracts or paradigms u_S.

These two interpretations of finite discrete probability theory extend easily to the case of point probabilities[5] p_j for $u_j \in U$, where $\Pr(S) = \sum_{u_j \in S} p_j$:

1. $(\rho(\Delta S))_{jj} = \chi_S(u_j)p_j / \Pr(S)$, so $\operatorname{tr}[P_{even}\rho(\Delta S)]$ = probability of getting *an* even-sided solid figure starting with the set of solid figures, and
2. $(\rho(S))_{j,k} = \chi_S(u_j)\chi_S(u_k)\sqrt{p_j p_k} / \Pr(S)$, so $\operatorname{tr}[P_{even}\rho(S)]$ = probability of getting '*the* even-sided solid figure' starting with '*the* solid figure.'

The whole of finite discrete probability theory can be developed in this manner, *mutatis mutandis*, for the #2 abstract paradigms instead of the usual #1 abstracts or events. That is the *paradigm interpretation* of probability theory and it gives a clear path to cross the bridge to QM.

11 Density matrices in Quantum Mechanics

This paradigm interpretation of finite probability leads directly to the use of probability in finite-dimensional quantum mechanics. The jump to quantum mechanics (QM) is to replace the reals $\sqrt{p_j p_k}$ in the density matrices by complex amplitudes. Instead of the set S represented by a column $|S\rangle$ of real 'amplitudes' $\sqrt{p_j}$, we have a normalized column $|\psi\rangle$ of complex numbers α_j whose absolute squares are probabilities: $|\alpha_j|^2 = p_j$, e.g.,

$$|S\rangle = \begin{bmatrix} 0 \\ \sqrt{p_2} \\ \sqrt{p_3} \\ \sqrt{p_4} \end{bmatrix} \rightsquigarrow |\psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

where $\alpha_1 = 0$ and $|\alpha_j|^2 = p_j$ for $j = 2, 3, 4$.

[5] Point probabilities are given by a probability density function $p : U \to [0,1]$ where $p(u_j) = p_j$ and $\sum p_j = 1$.

1. The *density matrix* $\rho(\Delta\psi)$ has the absolute squares $|\alpha_j|^2 = p_j$ laid out along the diagonal.
2. The *density matrix* $\rho(\psi) = |\psi\rangle\langle\psi|$ (where $\langle\psi|$ is the conjugate-tranpose$|\psi\rangle$) of has the j,k-entry as the product of α_j and α_k^* (complex conjugate of α_k), where $p_j = \alpha_j^*\alpha_j = |\alpha_j|^2$.

Thus:

$$\rho(\Delta\psi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \text{ and } \rho(\psi) = |\psi\rangle\langle\psi| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_2 & \alpha_2\alpha_3^* & \alpha_2\alpha_4^* \\ 0 & \alpha_3\alpha_2^* & p_3 & \alpha_3\alpha_4^* \\ 0 & \alpha_4\alpha_2^* & \alpha_4\alpha_3^* & p_4 \end{bmatrix}.$$

Some modern quantum mechanics texts, such as the Cohen-Tannoudji, Diu, and Laloë text [4, Vol. 1, p. 302] or the Auletta, Fortunato, and Parisi text [1], call attention to the special significance of the "coherences" represented by the non-zero off-diagonal terms.

> [The] off-diagonal terms of a density matrix...are often called *quantum coherences* because they are responsible for the interference effects typical of quantum mechanics that are absent in classical dynamics. [1, p. 177]

In the analogy between paradigm-universals in mathematics and superposition states in QM, *the point* is that an indefinite superposition QM state is a single entity that 'blobs out,' 'blurs out,' or renders indefinite the differences between the definite eigenstates in the (coherent) superposition. It is the indefiniteness aspect that carries over to the quantum case, not the classical notion of a 'paradigm.' The notion of #2 abstraction could be applied to any collections of distinct entities. For instance, in the example of three distinct properties $P(x)$, $Q(x)$, and $R(x)$ which could distinguish eight distinct elements, the property in common to the entities u_1, u_4, u_7 is not, in any useful sense, considered a 'paradigm' of anything. Moreover in the quantum case, it is not a zero-one affair whether two elements are equated as in the incidence matrices $\text{In}(S \times S)$ of the 'blobbed-out' sets; the off-diagonal elements in the density matrix give the 'amplitude' of the equating or cohering together of the eigenstates in the superposition state.

The classifying or measuring operation $\text{In}(\text{indit}(\pi)) \wedge \rho(\psi)$ could still be defined taking the minimum of corresponding entries in absolute value, but in QM it is obtained by what Auletta et al. [1, p. 279] call the *Lüders mixture operation.* If $\pi = \{B_1, ..., B_m\}$ is a partition according to the eigenvalues $\phi_1, ..., \phi_m$ on $U = \{u_1, ..., u_n\}$ (where U is an orthonormal basis set for the observable being measured), let P_{B_i} be the diagonal (projection) matrix with diagonal entries $(P_{B_i})_{jj} = \chi_{B_i}(u_j)$. Then $\text{In}(\text{indit}(\pi)) \wedge \rho(\psi)$ is obtained as:

$$\sum_{B_i \in \pi} P_{B_i} \rho(\psi) P_{B_i}$$

The Lüders mixture.

The probability of getting the result ϕ_i is:

$$\Pr(\phi_i|\psi) = \text{tr}[P_{B_i}\rho(\psi)].$$

These results are summarized in Table 5 (where $P_{|u_i\rangle}$ is the projection to the subspace generated by $|u_i\rangle$, and $P_{\{u_i\}}$ is the corresponding projection to the subset $\{u_i\}$).

Table 5	Probability theory	Quantum mechanics
Sample space	U = {u_1,...,u_n} set of outcomes	U = {u_1,...,u_n} orthonormal basis
#1 Abstract	Event = set of outcomes S ⊆ U	Equal mixture of eigenstates S
#2 Abstract	u_S = ⊞{u_i|u_i∈S} blob-sum of S	Equal superposition |ψ⟩ = (1/√|S|)Σ{|u_i⟩:u_i∈S}
#1 density matrix	Diagonal ρ(ΔS)$_{ii}$ = (1/|S|)χ_S(u_i)	ρ(Δψ) = (1/|S|)Σ{|u_i⟩⟨u_i|:u_i∈S} equal mixture of eigenstates of S
#2 density matrix	ρ(S)$_{ij}$ =(1/|S|)χ_S(u_i)χ_S(u_j)	ρ(ψ) = |ψ⟩⟨ψ| pure state of equal superposition of eigenstates of S
Non-degenerate measurement	$\sum_i P_{\{ui\}}$ρ(S)$P_{\{ui\}}$ = ρ(ΔS) Lüders mixture	$\sum_i P_{\|ui\rangle}$ρ(ψ)$P_{\|ui\rangle}$ = ρ(Δψ) Lüders mixture
Probability of u_i	tr[$P_{\{ui\}}$ρ(S)]	tr[$P_{\|ui\rangle}$ρ(ψ)]

Parallel operations in both versions of probability theory and quantum mechanics

12 Simplest Quantum Example

Consider a system with two spin-observable σ eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ (like electron spin up or down along the z-axis) where the given normalized superposition state is $|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle = \begin{bmatrix}\alpha_\uparrow\\ \alpha_\downarrow\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$ so the density matrix is $\rho(\psi) = \begin{bmatrix}p_\uparrow & \alpha_\uparrow\alpha_\downarrow^*\\ \alpha_\downarrow\alpha_\uparrow^* & p_\downarrow\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$ where $p_\uparrow = \alpha_\uparrow\alpha_\uparrow^*$ and $p_\downarrow = \alpha_\downarrow\alpha_\downarrow^*$. The measurement in that spin-observable σ goes from $\rho(\psi)$ to

$$\text{In}(\text{indit}(\sigma)) \wedge \rho(\psi) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \wedge \begin{bmatrix}p_\uparrow & \alpha_\uparrow\alpha_\downarrow^*\\ \alpha_\downarrow\alpha_\uparrow^* & p_\downarrow\end{bmatrix} = \begin{bmatrix}p_\uparrow & 0\\ 0 & p_\downarrow\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix} = \rho(\Delta\psi).$$

Or using the Lüders mixture operation:

$$P_\uparrow\rho(\psi)P_\uparrow + P_\downarrow\rho(\psi)P_\downarrow$$
$$= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}p_\uparrow & \alpha_\uparrow\alpha_\downarrow^*\\ \alpha_\downarrow\alpha_\uparrow^* & p_\downarrow\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}p_\uparrow & \alpha_\uparrow\alpha_\downarrow^*\\ \alpha_\downarrow\alpha_\uparrow^* & p_\downarrow\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}p_\uparrow & 0\\ 0 & p_\downarrow\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix} = \rho(\Delta\psi).$$

The two versions of $S = U$ give us two versions of finite discrete probability theory where: #1) U is the sample space or #2) u_U is *the* sample outcome.

1. The #1 classical version is the usual version which in this case is like flipping a fair coin and getting head or tails with equal probability (Figure 7)–like the mixed state:

$$\tfrac{1}{2}\left[|H\rangle\langle H| + |T\rangle\langle T|\right] = \rho(\Delta\psi) = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}.$$

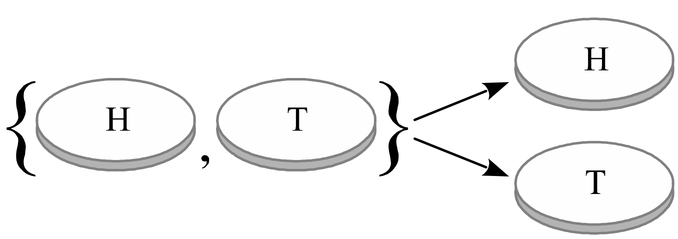

Figure 8: Outcome set for classical coin-flipping trial.

2. The #2 quantum version starts with the indefinite entity $u_U = \boxplus\{u_i \in U\}$, '*the* (indefinite) outcome', and a trial renders it into one of the definite outcomes u_i with some probability p_i so that u_U could be represented by the density matrix $\rho(U)$ where $(\rho(U))_{jk} = \sqrt{p_j p_k}$. In this case, this is like a coin $u_{\{H,T\}}$ with the difference between heads or tails rendered indefinite or blurred out, and the trial results in it sharpening to definitely heads or definitely tails with equal probability (Figure 8)–which in QM is the pure state (with the blobbed-out cross-terms $|H\rangle\langle T|$ and $|T\rangle\langle H|$ in the density matrix):

$$\frac{1}{\sqrt{2}}[|H\rangle + |T\rangle][\langle H| + \langle T|]\frac{1}{\sqrt{2}} = \rho(\psi) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

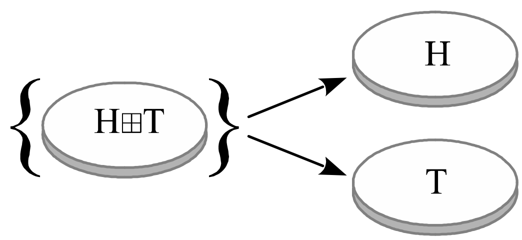

Figure 9: '*the* outcome state' for quantum coin-flipping trial.

Experimentally, it is not possible to distinguish between the #1 and #2 versions *by σ-measurements*–since, in either case, the result will be spin up or spin down (heads or tails) with equal probability. But in QM the two states $\rho(\Delta\psi)$ and $\rho(\psi)$ *can* be distinguished by measuring other observables like spin along a different axis as emphasized by Auletta et al. [1, p. 176]. Thus we *know* in QM which version is the superposition (pure) state $|\psi\rangle = \begin{bmatrix} \alpha_\uparrow \\ \alpha_\downarrow \end{bmatrix}$; it is *the #2 blob-state* $\rho(\psi)$.

13 Revisiting distinguishability of states

It is a standard doctrine of quantum information theory that one can always in principle distinguish two classical states like heads or tails, but cannot always distinguish two non-orthogonal pure states in quantum mechanics [17, p. 87] like $|0\rangle$ and $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$. But that changes when we admit blob- or paradigm-states like $H \boxplus T$, represented by $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, into our set of classical states–as in the paradigms version of finite discrete probability theory described above. Then $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ are both *pure* classical states and they cannot be reliably distinguished by any classification-measurements just like the quantum pure states $|0\rangle$ and $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$.

14 Conclusions

We have approached the paradigm interpretation of probability theory by starting with the logical situation of a universe U of distinct entities–where two distinct entities are always distinguished by some property as in Leibniz's principle of identity of indiscernibles. Given a property $S(x)$ on U, we can associate with it:

1. the #1 abstract object $S = \{u_i \in U | S(u_i)\}$, the set of $S(x)$-entities, or

2. the #2 abstract object $u_S = \boxplus \{u_i \in U | S(u_i)\}$ which is the abstract paradigm-entity expressing the properties common to the $S(x)$-entities but "abstracting away from," "rendering indefinite," "cohering together," or "blobbing or blurring out" the differences between those entities.

We argued that the mathematical machinery that could *distinctly* treat *both* abstractions was incidence matrices in logic and density matrices in probability theory:

1. #1 representation as $\text{In}(\Delta S)$ in logic or $\rho(\Delta S)$ in probability theory; and
2. #2 representation as $\text{In}(S \times S)$ in logic or $\rho(S)$ in probability theory.

This dove-tailed precisely into usual density-matrix mathematical treatment in QM of quantum states $|\psi\rangle$ as $\rho(\psi)$ which can be interpreted as *objectively* indefinite states, an interpretation of QM proposed by Abner Shimony.

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. [20, p. 47]
> But the mathematical formalism ... suggests a philosophical interpretation of quantum mechanics which I shall call "the Literal Interpretation." ...This is the interpretation resulting from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete.[21, pp. 6-7]

Yet since the ancient Greeks, we have the #2 Platonic notion of the abstract paradigm-universal '*the* S-entity', definite on what is common to the entities with the property $S()$, and indefinite on where they differ. By using incidence and density matrices to differentiate the #1 abstraction (e.g., set of distinct but parallel lines) and the #2 abstraction (e.g., *the* direction of the lines), we can cross the conceptual bridge to better understand indefiniteness in quantum mechanics by seeing the analogy:

The paradigm u_S, '*the* S-entity' represented by $\text{In}(S \times S) \approx$ the superposition state ψ represented by the density matrix $\rho(\psi)$.

This recalls Whitehead's quip that Western philosophy is "a series of footnotes to Plato." [25, p. 39]

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