On the Objective Indefiniteness Interpretation of Quantum Mechanics-rev

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Abstract

Classical physics and quantum physics suggest two different meta-physical notions of reality: the classical notion of a objectively definite reality "all the way down," and the quantum notion of an objectively indefinite type of reality. Part of the problem of interpreting quantum mechanics (QM) is the problem of making sense out of an objectively indefinite reality. Our sense-making strategy is to "follow the math" by showing that the mathematical way to describe indefiniteness is by partitions (quotient sets or equivalence relations). We follow the partitional math at both the levels of sets and complex vector spaces. We develop a pedagogical model of QM over sets or QM/sets so that the interpretive questions about QM can be split into two parts: QM/sets and full QM. The strategy is to first answer a question "in principle" in the simple setting of QM/sets, and then translate the answer into full QM. In particular, this approach is used in analyzing the notion of measurement. In both QM/sets and full QM, the mathematical machinery is shown to crucially involve partitions, and partitions are the mathematical way to describe indefiniteness. In this manner, following the math suggests that the realistic (i.e., non-epistemic) way to interpret quantum mechanics is, as emphasized by Abner Shimony, the objective indefiniteness interpretation.

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1 Introduction: Objective indefiniteness

The purpose of this paper is to describe an interpretation of standard Dirac-von-Neumann quantum mechanics (QM) that might be called the *objective indefiniteness interpretation.* There has long been the notion of subjective or epistemic indefiniteness ("cloud of ignorance") that is slowly cleared up with more discrimination and distinctions (as in the game of Twenty Questions). But the vision of reality that seems appropriate for quantum mechanics is objective or ontological indefiniteness. The notion of objective indefiniteness in QM has been most emphasized by Abner Shimony ([39], [40], [41]).

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. [39, p. 47]
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> The fact that in any pure quantum state there are physical quantities that are not assigned sharp values will then mean that there is *objective indefiniteness* of these quantities. [41, p. 27]

Shimony also suggested that this interpretation of QM could be called the "Literal" interpretation.

> These statements ... may collectively be called "the Literal Interpretation" of quantum mechanics. This is the interpretation resulting from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete.[42, pp. 6-7]

The view that a description of a superposition quantum state is a *complete* description means that the indefiniteness of a superposition state is objective or ontological and not just subjective or epistemological.

In addition to Shimony's "objective indefiniteness" (the phrase also used by Gregg Jaeger [24] and used here), other philosophers of physics have suggested related ideas such as:

- Peter Mittelstaedt's "incompletely determined" quantum states with "objective indeterminateness" [35],
- Paul Busch and Gregg Jaeger's "unsharp quantum reality" [4],
- Paul Feyerabend's "inherent indefiniteness" [16],
- Allen Stairs' "value indefiniteness" and "disjunctive facts" [43],
- E. J. Lowe's "vague identity" and "indeterminacy" that is "ontic" [31],
- Steven French and Decio Krause's "ontic vagueness" [20],
- Paul Teller's "relational holism" [45], and so forth.

Among the generation of "classical" mathematicians and physicists who developed quantum mechanics, we will see that Hermann Weyl and Werner Heisenberg came the closest to the objective indefiniteness interpretation.

Today, the idea that a quantum state is, in some sense, indefinite, blurred, or like a cloud is now rather commonplace even in the popular literature. Much of the literature on the interpretation of QM represents attempts to escape the standard Dirac-Von-Neumann QM in various flights of fancy (many worlds, hidden variables, etc.), while the Literal or objective indefiniteness interpretation outlined here is based on trying to directly making sense out a superposition as a complete description of an objectively indefinite, "blurred," or "cloud-like" state.

2 How to mathematically describe indefiniteness: partitions

2.1 Two notions of reality

There are two very different (and, in a mathematical sense, dual) notions of reality:

Classical-definite the common-sense notion of objectively definite reality assumed in classical physics, and

Quantum-indefinite the notion of objectively indefinite reality suggested by quantum physics.

There are a few intuitive metaphors that might be used to help fix ideas. When the police try to obtain a description of a face from a witness, there are two different methods. One method is to go through a book of pictures of definite faces or mugshots–which is the classical-definite way to arrive at a definite face. The other method is to start with a blank police sketch of a face and then to fill in the various features to arrive at a definite face–which is a metaphor for the quantum-indefinite approach. For a classical-definite description of a change of states, one metaphor is a motion picture that goes from a definite frame to definite frame. For a quantum-indefinite description of a change of states, a metaphor is to go from a definite state to an indefinite one, like a camera going out of focus, and then returning to a definite state (measurement), like a camera coming into focus.

Our common-sense notion of reality is one of definiteness, so how can we make sense out of the notion of indefiniteness? If there is some basic duality between these two types of reality, then it should show up in mathematics, and so it does. The duality is the "turn-around-the-arrows" duality of category theory [33]. This duality can be expressed in the duality between subsets and quotient sets (or equivalence relations or partitions), and thus the classical Boolean logic of subsets has a "classical" dual in the logic of partitions ([12]; [14]). Within category theory, this duality gives the two dual types of universal constructions: limits and colimits (where the prefix "co-" is often used for the dual concepts obtained by taking a quotient to describe indefiniteness as the result of factoring out some definiteness). Hence our basic strategy to describe these two dual notions of reality (and particularly the hard-to-understand quantum-indefinite notion) is to *follow the math*.

We will follow the math at two different levels: at the relatively simple level of sets (where things are more "clear and distinct"), and at the more sophisticated level of complex vector spaces (e.g., as used in the Hilbert spaces of QM). This two-fold expository strategy involves developing a pedagogical or "toy" model of QM, *quantum mechanics over sets* (QM/sets), where many of the points can be made in the particularly simple setting of sets. The use of a rich pedagogical model of QM allows interpretive or "philosophical" questions about QM to be addressed in two stages:

1. the first stage where the question might be answered "in principle" in terms of sets in the pedagogical model of QM/sets, and

2. the second stage where the answer using the simplicities of QM/sets is translated into the complexities of full QM.

Since so many quantum controversies represent disagreements "in principle," this first stage provides a new approach to some old questions–like the question of measurement.

2.2 Partitions factor out definiteness to describe indefiniteness

How can indefiniteness be depicted mathematically? In a word, partitions. The basic idea is to start with what is taken as the full range of definite or "eigen" alternatives and then factor or quotient out the surplus definiteness to obtain an equivalence relation or partition.

Starting with some universe set U of fully distinct and definite elements, a *partition* $\pi = \{B_i\}$ (i.e., a set of disjoint blocks $B_i \subseteq U$ whose union is U) collects together in a block (or cell) B_i the distinct elements $u \in U$ whose distinctness is to be ignored or factored out, but the blocks are still distinct from each other. Each block is an equivalence class in the associated equivalence relation on U. Each block-in-a-partition is to be interpreted not as a set of definite entities but as a single indefinite "superposition" entity that is indefinite between the potential definite entities contained in it as a subset.

The two different notions of reality correspond to two different ways to interpret such a simple notion as a subset $\{a, c\} \subseteq U$:

1. the usual common-sense notion of a set $\{a, c\}$ as a collection of two different definite entities, or

2. the "quantum" or objectively indefinite interpretation of $\{a, c\}$ as a single "superposition" entity (e.g., in QM/sets) indefinite between the distinct (eigen-)possibilities $\{a\}$ or $\{c\}$.

Consider a purely combinatorial example of how partitions factor out definiteness: the calculation of the binomial coefficient $\binom{N}{m} = \frac{N!}{m!(N-m)!}$. The idea is to count the number of m-ary subsets of an N-ary set ($m \leq N$) where the different orderings of the otherwise same m-element subset and the same $N-m$-element subset are surplus definiteness that needs to be factored out. The method of calculation is to first count the number of possible *orderings* of the whole N-ary subset which is $N! = N(N-1)\ldots(2)(1)$. Then we want to quotient out the cases that are distinct only because of different orderings. There are $m!$ ways to permute the first m elements in the given ordering–leaving the last $N-m$ elements the same. Thus we take the first quotient by identifying any two of the $N!$ different orderings if they differ only in a permutation of those first m elements. Since there are $m!$ such permutations, there are now $N!/m!$ equivalence classes or blocks in the resulting partition of the $N!$ orderings. But these equivalence classes still count as distinct the different orderings of the last $N-m$ elements so we further identify blocks which just have a permutation of the last $N-m$ elements to make larger blocks. Then the result is $\binom{N}{m} = \frac{N!}{m!(N-m)!}$ blocks in the partition which is the number of m-element subsets (which equals the number of $N-m$-element subsets) out of an N-element set.

In this example, the set of fully determinate alternatives are the $N!$ orderings of the N-element set. Then to consider the subsets of cardinality m (and thus the complementary subsets of cardinality $N-m$), we must quotient out the number of possible orderings $m!$ and $(N-m)!$ to render the ordering of the elements in the subsets indefinite or indeterminate as to their ordering.

Example 1 *To be concrete, consider a set* $\{1,2,3,4\}$ *of* $N=4$ *numerals so the universe of fully distinct orderings has* $4! = 24$ *elements* $\{1234, 1243, \ldots\}$*. How many* 2*-element subsets are there? The first quotient groups together or identifies the orderings which only permute the first* $m=2$ *elements so two of the blocks in that partition are* $\{1234, 2134\}$ *and* $\{1243, 2143\}$*, and there are* $N!/m! = 24/2 = 12$ *such blocks. Each block has the same final* $N-m=2$ *elements in the ordering so we further identify the blocks that differ only in a permutation of those last* $N-m$ *elements. One of the blocks in that final partition is* $\{1234, 2134, 1243, 2143\}$ *and there are* $\frac{N!}{m!(N-m)!} = \frac{24}{(2)(2)} = 6$ *such blocks with four elements in each block. Each block is distinct from the other blocks in the first* m *elements and in the last* $N-m$ *elements of the orderings in the block so the block count is just the number of subsets of* m *elements (which equals the number of subsets of* $N-m$ *elements as well) where each block is indefinite as to the ordering of elements within the first* m *elements and within the last* $N-m$ *elements. This could be illustrated as follows where, for instance, the block* $\{1234, 2134\}$ *is indicated as the "superposition"* $1234 + 2134$ *in anticipation of the treatment in QM/sets.*

Figure 1: Introducing ordering and then factoring it out
to make the blocks indefinite as to ordering

Hermann Weyl makes the same point using an example slightly more complicated than the binomial coefficient. He starts with a set or "aggregate S ... of elements each of which is in a definite state" [47, p. 239] and then considers a partition or equivalence relation whose k blocks or classes $C_1, \ldots, C_k$ can be thought of as boxes into which the n elements of S may be placed (some boxes might be empty).

> A definite *individual state* of the aggregate S is then given if it is known, for each of the n marks p [DE: which distinguish the n elements of S], to which of the k classes [or boxes] the element marked p belongs. Thus there are k^n possible *individual states* of S. If, however, no artificial differences between elements are introduced by their labels p and merely the intrinsic differences of state are made use of, then the aggregate is completely characterized by assigning to each class C_i $(i = 1, ..., k)$ the number n_i of elements of S that belong to C_i. [47, p. 239]

Those occupation numbers n_i would then characterize "the *visible* or *effective state* of the system S." [47, p. 239] Thus Weyl points out that the mathematical treatment of indefiniteness starts with the definiteness given here by the markings p on the n elements distributed between the k boxes C_i, and then one erases or quotients out the markings p so the blocks or boxes in the partition have only an occupation number n_i with no distinctions between the n_i elements in each box C_i. When this scheme for representing indefiniteness is applied in quantum mechanics, then it is an objective indefiniteness in that no further differentiation between the elements of a box C_i is possible.

> Since photons come into being and disappear, are emitted and absorbed, they are individuals without identity. No specification beyond what was previously called the effective state of the aggregate is therefore possible. Hence the state of a photon gas is known when for each possible state α of a photon the number n_α of photons in that state is given (Bose-Einstein statistics of radiation). [47, p. 246]

Thus within QM, the treatment of the indefiniteness due to the indistinguishability of quantum particles of the same type is to artificially treat them as distinct and then collect together (like collecting together in a block of a partition) or superpose the permutations of the particles (in a totally symmetric or totally antisymmetric manner) that factors out their supposed distinctness (see any QM text such as [8, Chapter XIV]) to get a representation of an objectively indefinite state.

Our point is that indefiniteness is mathematically described–using sets–by taking a partition or quotient of a set of definite entities. Each block of a partition is indefinite between the elements within it, but the blocks of the partition are distinct from one another. In full QM, indefiniteness is represented in the corresponding way but with the superposition being the sum of certain definite eigen-states in a Hilbert space.

2.3 The common sense notion of definiteness

In the common-sense classical view of reality, it is definiteness (rather than turtles) "all the way down." Every entity or thing definitely has a property P or definitely has the property $\neg P$. Peter Mittelstaedt quotes Immanuel Kant's treatment of the idea of complete determinateness:

> Every thing as regards its possibility is likewise subject to the principle of complete determination according to which if all possible predicates are taken together with the contradictory opposites, then one of each pair of contradictory opposites must belong to it. (Kant quoted in: [35, p. 170])

Given a universe set U, a predicate P is represented by the subset $S \subseteq U$ of elements that have the property, and the complement subset $S^c = U - S$ represents the elements that have the property $\neg P$. Moreover an element $u \in U$ has properties "all the way down" so that it is uniquely determined by all the subsets S containing u as in Leibniz's principle of the identity of indiscernibles [28].

2.4 The duality of subsets and partitions

These two contrasting views of reality can be represented mathematically using the classical Boolean lattice of subsets and the "dual" lattice of partitions. In the lattice of subsets, the partial order is subset inclusion. In the lattice of partitions, the partial ordering is refinement where given two partitions $\pi = \{B\}$ and $\sigma = \{C\}$ on a universe set U, the partition π refines σ (written $\sigma \preceq \pi$) if for every block $B \in \pi$ of π, there is a block $C \in \sigma$ of σ such that $B \subseteq C$. Consider the simple case of a universe $U = \{a, b, c\}$ of three elements giving the three fully-distinct blocks $\{a\}$, $\{b\}$, and $\{c\}$ (rather than the 24 distinct possibilities in the previous figure).

Figure 2: The two "dual" lattices of subsets and partitions

In the lattice of partitions, a block $\{a\} + \{c\} = \{a, c\}$ represented an entity that is indefinite between $\{a\}$ and $\{c\}$ but distinct from $\{b\}$ (just as in the previous example, 1234+2134 was indefinite between ordering of the first two elements but was distinct from, say, $1243 + 2143$). Pascual Jordan used tongue-in-cheek anthropomorphic terms to describe the impact of a spatial measurement on a spatial indefinite electron:

> the electron is forced to a decision. We compel it to assume a definite position; previously, in general, it was neither here nor there; it had not yet made its decision for a definite position... . (Quoted in: [27, p. 161])

In the Boolean lattice of subsets, a subset $\{a, c\}$ is simply a collection of two fully determinate elements; it is not a single entity that has "not yet made its decision" to be $\{a\}$ or $\{c\}$.

Subsets and quotient sets (or partitions) are mathematically *dual* concepts in the reverse-the-arrows sense of category theory, e.g., a subset is the direct image of a set monomorphism while a set partition is the inverse image of an epimorphism. This duality is familiar in abstract algebra in the interplay of subobjects (e.g., subgroups, subrings, etc.) and quotient objects (e.g., quotient groups, quotient rings, etc.). William Lawvere calls the general category-theoretic notion of a subobject a *part*, and then he notes: "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition."[30, p. 85]

The logic appropriate for the usual notion of fully definite reality is the ordinary Boolean logic of subsets [2] (usually mis-specified as the special case of "propositional" logic). We have seen that the other vision of objectively indefinite reality suggested by QM is mathematically described by the (equivalent) concepts of quotient sets, partitions, or equivalence relations. There is an equally fundamental logic of those quotient sets, equivalence relations, or partitions ([12] and [14]). The two logics of the dual concepts might be associated with the two visions of reality: (1) the classical common-sense notion of elements determined "all the way down" and (2) the quantum notion of indefinite "superposition" entities represented by the blocks in partitions.

3 How to intuitively visualize indefiniteness

3.1 Perches and flights

There is no pretension that we have a clear and distinct mental image of an indistinct objective state. But that does not prevent one from trying to build *some* imagery no matter how inadequate to our common sense.

In Boole's logic of subsets, each element u of the universe set U either definitely has or does not have a given property P (represented as a subset S of the universe). Change takes place by the definite properties changing–like a slide show going from one fully definite image to another or a motion picture going from one frame to another. For a hound to go from point A to point B, there must be some trajectory of definite ground locations from A to B. One might be subjectively or epistemologically indefinite about the exact positions along the hound's path even though the path is objectively definite.

In the dual case of partitions, a partition $\pi = \{B\}$ is made up of disjoint blocks B whose union is the universe set U. The blocks in a partition have been distinguished from each other by the partition, but the elements within each block have not been distinguished from each other; instead they are identified by the associated equivalence relation. Each block B represents the objectively indefinite (pure) state represented by superposing the definite singletons $\{u\} \subseteq B$. When more distinctions are made (in the QM/sets-version of a measurement considered below), the blocks get smaller and the partitions (= QM/sets-version of mixed states) become more refined until the discrete partition $\mathbf{1} = \{\{u\} : \{u\} \subseteq U\}$ is reached where each block is a singleton (= the QM/sets-version of a non-degenerate measurement yielding a completely decoherent mixed state). Change takes place by evolving from a definite state to an indefinite state and then back to another definite state–like a slide show where the transition is from an in-focus picture going out-of-focus and then refocusing on another in-focus picture, or like a camera going out of focus and then back into focus. For a hawk, as opposed to a hound, to go from point A to point B, it would go from a definite perch at A into a flight of indefinite ground locations, and then would have a definite perch again at B (like the quantum trajectory of a particle in a cloud chamber).[1]

Figure 3: How a hound and a hawk go from A to B

The imagery of having a sharp focus versus being out-of-focus could be used if one is clear that it is the reality itself that is in-focus (definite) or out-of-focus (indefinite), not just the image through, say, a camera or microscope. A classical trajectory is like a moving picture or slide show of changing

[1]The "flights and perchings" metaphor is from William James [25, p. 158] and according to Max Jammer, that description "was one of the major factors which influenced, wittingly or unwittingly, Bohr's formation of new conceptions in physics." [26, p. 178] The hawks and hounds pairing comes from Shakespeare's Sonnet 91.

sharp or definite in-focus realities, whereas the quantum trajectory starts with a sharply focused reality, goes out of focus, and then returns to an in-focus reality (e.g., by a measurement).

In the objective indefiniteness interpretation, a subset $S \subseteq U$ of a universe set U should be thought of as a single indefinite entity S that is *represented* as the superposition of the definite entities $\{u\} \subseteq S$–just as a single superposition vector is represented as a weighted vector sum of certain basis eigenvectors ("eigen" should be translated as "definite" here).

3.2 Heisenberg on potentialities

Abner Shimony ([39] and [40]), in his description of a superposition state as being objectively indefinite, sometimes used Heisenberg's [22] language of "potentiality" and "actuality" to describe the relationship of the eigenvectors that are superposed to give an objectively indefinite state.[2] This terminology could be adapted to the case of the sets. The singletons $\{u\} \subseteq S$ are "potential" in the objectively indefinite superposition S, and, with further distinctions, the indefinite entity S might "actualize" to $\{u\}$ for one of the "potential" $\{u\} \subseteq S$. Starting with S, the other $\{u\} \nsubseteq S$ (i.e., $u \notin S$) are not "potentialities" that could be "actualized" with further distinctions.

This terminology is, however, somewhat misleading since the indefinite entity S is perfectly actual (in the objectively indefinite interpretation); it is only the multiple eigenstates $\{u\} \subseteq S$ that are "potential" until "actualized" by some further distinctions. A (non-degenerate) measurement is not a process of a potential entity (existing in some ontological limbo) becoming an actual entity; it is a process of an actual *indefinite* entity becomes an actual definite entity. Since a distinction-creating measurement goes from an actual indefinite entity to an actual definite entity, the potential-to-actual language of Heisenberg should only be used with proper care–if at all.

3.3 Heisenberg on substance and form

Heisenberg liked to clothe his more metaphysical ideas in discussions of ancient Greek philosophy. In that spirit, the conceptual duality between the lattice of subsets and the lattice of partitions could be described using Heisenberg's rather meta-physical notions of *substance*[3] and *form* (as in in-formation). For each lattice where $U = \{a, b, c\}$, the movement from bottom to top can be described in terms of substance and form.

[2] See the paper by Gregg Jaeger and Paul Busch in this Volume.

[3] Heisenberg identifies "substance" with energy.

> Energy is in fact the substance from which all elementary particles, all atoms and therefore all things are made, and energy is that which moves. Energy is a substance, since its total amount does not change, and the elementary particles can actually be made from this substance as is seen in many experiments on the creation of elementary particles. [22, p. 63]

Figure 4: Conceptual duality between the subset and partition logics

At the bottom of the Boolean lattice is the empty set $\emptyset$ which represents no substance. As one moves up the lattice, new elements of substance always with fully definite properties are created until finally one reaches the top, the universe U. Thus new substance is created in moving up the lattice but each element is already fully formed.

At the bottom of the partition lattice is the indiscrete partition or "blob" $\mathbf{0} = \{U\}$ (where the universe set U makes one block) which represents all the substance but with no distinctions to in-form the substance.[4] As one moves up the lattice, no new substance is created but distinctions objectively in-form the indistinct elements as they become more and more distinct, until one finally reaches the top, the discrete partition $\mathbf{1}$, where all the eigen-elements of U have been fully distinguished from each other.[5] It was previously noted that a partition combines indefiniteness (within blocks) and definiteness (between blocks). At the top of the partition lattice, the discrete partition $\mathbf{1} = \{\{u\} : \{u\} \subseteq U\}$ is the result making all the distinctions to eliminate the indefiniteness. Thus one ends up at the "same" place (macro-universe of distinguished elements) either way, but by two totally different but dual ways.[6]

3.4 A superposition is *not* like a double-exposure photograph

Consider a partition $\pi = \{\{b\}, \{a, c\}\}$ on the three-element universe $U = \{a, b, c\}$. The block $S = \{a, c\}$ is objectively indefinite between $\{a\}$ and $\{c\}$. This objective indefiniteness of $\{a, c\}$ is *not* well-described as saying that indefinite pre-distinction entity is "simultaneously both $\{a\}$and $\{c\}$" (like the common misdescription of the undetected particle "going through both slits" in the double-slit experiment). It is a misdescription like saying that the $45°$ unit vector $(1, 1)/\sqrt{2}$ on the real x, y-plane is simultaneously on the x-axis and on the y-axis. A superposition of two sharp eigen-alternatives should *not* be thought of like a double-exposure photograph (a relic of film cameras) which has two fully definite images (e.g., simultaneously a picture of say $\{a\}$ and $\{c\}$). Instead of a double-exposure photograph, the superposition should be thought of as *representing* or describing one indefinite reality that with further distinctions could sharpen to either of the sharp realities

[4] The "blob" is the set-version of a pure state in QM prior to a distinctions-creating measurement that decoheres the pure state into non-blob partitions analogous to a mixed state. See the treatment below of measurement in terms of density matrices.

[5] This notion of logical in-formation as distinctions is based on partition logic just as logical probability is based on subset logic ([11] and [13]). That is, the *logical entropy* of a partition is the normalized counting measure of the distinctions (ordered pairs of elements in different blocks) of partitions on U just as the Laplace-Boole *logical probability* of a subset is the normalized counting measure of the elements in the subsets (events) of the finite universe set U (set of equiprobable outcomes).

[6] In treating the universe $U = \{u_1, u_2, ..., u_n\}$ and the discrete partition $\mathbf{1} = \{\{u_1\}, \{u_2\}, ..., \{u_n\}\}$ as the "same" we are neglecting the distinction between u and $\{u\}$ for $u \in U$.

(mathematically, the distinctions project the 45° unit vector to either the x or y axis). That is, there must be some way to indicate which definite realities could be obtained by making further distinctions (measurements), and *that* is why the blurred or cloud-like indefinite reality is *represented* by mathematically superposing the definite possibilities.

Instead of a double-exposure photograph, a superposition representation might be thought of as ”a photograph of clouds or patches of fog.” (Schrödinger quoted in: [19, p. 66]) Schrödinger distinguishes a ”photograph of clouds” from a blurry or cloudy photograph presumably because the latter might imply that it was only the photograph that was blurry or cloudy while the underlying objective reality was sharp. The ”photograph of clouds” imagery for a superposition connotes a clear and complete photograph of an objectively ”cloud-like” or indefinite reality.

Regardless of the (imperfect) imagery, one needs some way to indicate what are the definite eigenstates that could be ”actualized” from a single indefinite entity S, and *that* is the role of conceptualizing a subset S as ”superposing” certain ”potential” eigenstates, i.e., the singletons $\{u\} \subseteq S$.

This crucial point might be illustrated using some Guy Fawkes masks (a variation on the police sketch-pad metaphor). Suppose there are two ”orthogonal” eigenstates of having a goatee or a mustache, Mask 1 and Mask 2, represented formally by $|goatee\rangle$ and $|mustache\rangle$.

Figure 5: Objectively indefinite Mask 3 (not Mask 4) *represented* by superposition of distinct eigen-alternatives $|goatee\rangle + |mustache\rangle$

The objectively indefinite state is the Mask 3 without facial hair distinctions of goatee or mustache, but it is formally *represented* as the addition or superposition $|goatee\rangle + |mustache\rangle$ of the possible definite states.[7] That superposition is unfortunately usually misinterpreted as representing the double-exposure Mask 4 which, like the ”particle going through both slits,” is (by orthogonality) an impossible state.

[7] The ”measurement” of going from the indefinite Mask 3 to either of the definite Masks 1 or 2 could be seen as a special case of the process of a police artist going from essentially a blank slate to a definite sketch of the suspected criminal.

3.5 Where is a spatially-indefinite particle?

Under this objectively-indefinite way of interpreting the "wave function" or state vector formalism, much of the literature on interpreting the "wave function," not to mention the imagery of an electron going through both slits or a photon going through both arms of an interferometer, is wrong-footed from the beginning. The difficult imagery lies in imagining an objectively indefinite state, particularly when we try to force it into the space of definite or eigen states (like trying to locate Mask 3 in a space consisting of two definite states, Mask 1 and Mask 2).

In this whole vision of reality as having indefinite or out-of-focus objective states, the "space" of fully definite eigenstates is *not* the only reality; there are the other possible indefinite superposition states that could be just as real. The most common interpretive error is to force a quantum state into the "space" of definite states. For instance in the double-slit experiment, the possibilities are not exhausted by saying the particle went through slit 1, through slit 2, or (impossibly) through both slits. In the unmeasured case, the particle has a spatial mode of being indefinite between slit 1 and slit 2 that may be represented by the superposition $|slit1\rangle + |slit2\rangle$ which is not a definite spatial location. Using the previous example of a partition lattice with $\{a\} = |slit1\rangle$ and $\{c\} = |slit2\rangle$, we can answer the "Where's Waldo" question for the unmeasured particle in the spatially indefinite mode $\{a, c\}$.

Figure 6: If $\{a\} = |slit1\rangle$ and $\{c\} = |slit2\rangle$,
what is the state of the undetected particle that "goes through both slits"?

Stepping back for a moment, it might be noted that we are answering the question of "where is the spatially indefinite particle?" using sets in the pedagogical model of QM/sets (to be treated more formally below). In full QM, the set superposition $\{a, c\}$ would be rendered as something like $[|slit1\rangle + |slit2\rangle] / \sqrt{2}$ in a Hilbert space, but the point remains that it is not to be found among the eigenstates of a spatial observable. This is an example of addressing an interpretive or "philosophical" question in two parts; the first part addressed in QM/sets and then the second part translates the answer into full QM.

3.6 Applying the no-double-exposure point to dynamics

The most important consequence of the no-double-exposure interpretation is in better understanding quantum dynamics without measurement. Since the static objectively indefinite states are *represented* by the linear superposition of the possible definite states, the linear dynamic evolution of the indefinite states is thus *represented* as the linear superposition of the evolution of the definite states. The double- or multiple-exposure misinterpretation of the linear evolution of a superposition state is the source of the usual wave imagery in QM (e.g., as in Fourier analysis). But the point is that the evolving "wave function" or state vector as a superposition of evolving eigenstates, is only the way to *describe* the evolution of the one *indefinite state* that is indefinite between those evolving eigenstates. Since the indefinite state is not actually the (impossible) "multiple exposure" of actual orthogonal definite states, the usual wave imagery of superposition and interference of separate waves, as if the "wave function" represented actual waves of some sort, is rather misleading. The superposition and "interference" of evolving possible definite states is just how to *represent* or describe the evolution of objectively indefinite state that is indefinite between those evolving definite possibilities.

Many quantum theorists have long argued, correctly in our opinion, that the "wave function" (or state vector) should not be interpreted as describing actual waves that interfere in some higher dimensional space, i.e., against what we have called the "multiple-exposure" misinterpretation of a superposition state in full QM. On the objective indefiniteness interpretation, the realistic interpretation of a superposition state vector (or "wave function") is as an objective state indefinite between the potential alternatives, and the evolution of the indefinite state would *thus* be mathematically described as the superposition of the evolving potential alternatives.[8]

4 Whence partitions? Two ways to define partitions

4.1 Set partitions from set attributes

To recapitulate, indefiniteness is represented mathematically by partitions. Take the set of all definite possibilities, and then quotient out some definiteness to obtain the blocks in a partition (or, equivalently, equivalence classes of an equivalence relation).

There are two opposite ways to specify the partitions: the top-down way (using an attribute or observable) and the bottom-up way (using a symmetry group). And each way can be illustrated in the relatively simple setting of sets and in the more "complex" setting of complex vector spaces. In the latter setting, the partitional mathematics describing indefiniteness gives the essential structure of QM–which shows how the objective indefiniteness interpretation QM is suggested by "following the math."

We start with the top-down method of defining partitions using an attribute defined on a set. Take the universe set as some specific set of people, say, in a room. People have numerical attributes like weight, height, or age as well as non-numerical attributes with other values such place of birth, family name, and country of citizenship. Abstractly an *attribute on a universe set* U is a function $f : U \to R$ from U to some set of values R (usually the reals $\mathbb{R}$). In subset logic, an element $u \in U$ either has a *property* represented by a subset $S \subseteq U$ or not; in partition logic, an attribute f assigns a value $f(u)$ to each $\{u\} \subseteq U$. The two concepts of a property and an attribute overlap for binary attributes where the attribute might be represented by the characteristic function $\chi_S : U \to 2 = \{0, 1\}$ of a subset $S \subseteq U$.[9]

Each attribute $f : U \to R$ on a universe U determines the *inverse-image partition* $f^{-1} =$

[8] This point can be made pedagogically using the QM/sets model developed later in the paper. State vectors are subsets that add mod(2) so that $\{a, b\} + \{b, c\} = \{a, c\}$ with the $\{b\}$'s "interfering" and cancelling out. The dynamics in QM/sets is by non-singular linear transformations in vector spaces over $\mathbb{Z}_2$. For instance, the basis $\{a\}$, $\{b\}$, $\{c\}$ in $\wp(\{a, b, c\}) \cong \mathbb{Z}_2^3$ could evolve in a non-singular manner as follows: $\{a\} \to \{a, b\}$, $\{b\} \to \{b, c\}$, and $\{c\} \to \{a, b, c\}$. Then the superposition or indefinite state $\{a, b\}$ linearly evolves to $\{a, b\} + \{b, c\} = \{a, c\}$. That is a description of *how indefinite states evolve*, not a description of how a definite entity denoted by $+\{b\}$ interferes with and cancels the definite entity denoted by $-\{b\}$ [since $-1 = +1$ mod(2)]–like the crest of one water wave cancelling the trough in another water wave. This all follows from the original reinterpretation of a set $\{a, b\}$ not as a collection of two definite entities but as a single entity indefinite between $\{a\}$ and $\{b\}$.

[9] To be technically precise, a subset S is given by a binary attribute $\chi_S : U \to 2 = \{0, 1\}$ *plus* the designation of an element $1 \in 2$ so that $S = \chi_S^{-1}(1)$ as in Lawvere's well-known subobject-classifier diagram [30, p. 39].

$\{f^{-1}(r) \neq \emptyset : r \in R\}$. Attributes are one way to define a partition on a set U. Since this method of defining a partition starts with a numerical attribute $f(u)$ already assigned to each $u \in U$, it may be called the *top-down* method.

4.2 Set partitions from set representations of groups

Another more "bottom-up" way to define a partition on U is to map the elements $u \in U$ to "similar" elements u' by some set of transformations $G = \{t : U \to U\}$. This defines a binary relation: uGu' if there exists a $t \in G$ such that $t(u) = u'$. In order to define a partition, the binary relation uGu' has to be an equivalence relation so the blocks of the partition are the equivalence classes. The three requirements for an equivalence relation are reflexivity, symmetry, and transitivity.

- For the relation to be reflexive, i.e., uGu for all $u \in U$, it is sufficient for the set of transformations G to contain the identity transformation $1_U : U \to U$.
- For the relation to be symmetric, i.e., uGu' implies $u'Gu$, it is sufficient for each $t \in G$ to have an inverse $t^{-1} \in G$ where $U \xrightarrow{t} U \xrightarrow{t^{-1}} U = 1_U = U \xrightarrow{t^{-1}} U \xrightarrow{t} U$.
- For the relation to be transitive, i.e., uGu' and $u'Gu''$ imply uGu'', it is sufficient for each $t, t' \in G$ that $t't : U \xrightarrow{t} U \xrightarrow{t'} U$ is also in G.

These three conditions, the existence of the identity, the existence of an inverse, and closure under composition, define a *transformation group* $G = \{t : U \to U\}$, i.e., a *group action* on a set U. Equivalently, a *set representation* of a group G is given by a group homomorphism $T : G \to S(U)$, where $S(U)$ is the symmetric group of all permutations t of the set U. An *abstract group* satisfies these three conditions where the composition is also required to be associative in the sense that for any $t, t', t'' \in G$, $(t''t')\,t = t''\,(t't)$. For a transformation group, the composition is automatically associative.

This connection between groups and equivalence relations is, of course, well-known [5], and is probably as old as the notion of a group. Simply put, a (transformation) group is a set of "actions" on a set that define an equivalence relation. Instead of elements $u, u' \in U$ being collected in the same block by having the same attribute value $f(u) = f(u')$, the group transformations take any element u to a "similar" or "symmetric" element $t(u) = u'$. A subset $S \subseteq U$ is *invariant under* G if for any $t \in G$, $t(S) \subseteq S$. A minimal invariant subset is an *orbit*, and the set partition defined by the transformation group G is the *partition of orbits*. That is the *bottom-up* method of defining a set partition since we don't begin with some attribute-value already assigned to the elements of U.

What is the significance of the blocks in the partition of minimal invariant subsets? Often the treatment of symmetry groups focuses on what is invariant or conserved, e.g., the perspective of Noether's theorem [3].

There is *another* perspective with which to view the orbits of a symmetry group. To represent an indefinite reality, there is first some notion of the fully definite eigen-alternatives that are then superposed to represent something indefinite between those alternatives.

What determines the set of definite eigen-alternatives?

Given a set of symmetries on a set, in what different ways can there be distinct subsets that still satisfy the constraints of the symmetry operations? The minimal invariant subsets or orbits of a set representation of a symmetry group provide the answer to that question about the variety of "atomic" eigen-forms consistent with the symmetries.

This question and the answer become more significant when we move beyond structure-less sets to complex vector spaces. The minimal invariant subsets, the orbits, then become the minimal invariant subspaces, the irreducible subspaces, which are the carriers of the important irreducible representations or irreps in vector space representations of groups.

4.3 Set partitions from other set partitions

Since there are two ways to make set partitions, from attributes and from symmetry groups, there are also two ways to refine a partition by making distinctions, "joining" an attribute with another (compatible) attribute or breaking symmetries by moving to a subgroup of the symmetry group. Since our eventual focus is on measurement, we will elaborate on making distinctions by the joining of attributes.[10] What is the mathematical operation for making the distinctions (as in a measurement)? It is the *join operation* from partition logic. But before two set partitions can be joined to form a more refined partition with more distinctions, they must be *compatible* in the sense of being defined on the same universe set. If two set partitions $\pi = \{B\}$ and $\sigma = \{C\}$ are compatible, i.e., are partitions of the same universe U, then their *join* $\pi \vee \sigma$ is the set partition whose blocks are the non-empty intersections $B \cap C$.

Since two set attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$ define two inverse image partitions $\left\{f^{-1}(r)\right\}$ and $\left\{g^{-1}(s)\right\}$ on their domains, we need to extend the concept of compatible partitions to the attributes that define the partitions. That is, two attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$ are *compatible* if they have the same domain $U = U'$.

Given two compatible set attributes $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$, the join of their inverse-image partitions has as blocks the non-empty intersections $f^{-1}(r) \cap g^{-1}(s)$. Each block in the join could be characterized by the ordered pair of "eigenvalues" (r, s). An "eigenvector" $S \subseteq U$ of f, i.e., $S \subseteq f^{-1}(r)$, and of g, i.e., $S \subseteq g^{-1}(s)$, would be a "simultaneous eigenvector": $S \subseteq f^{-1}(r) \cap g^{-1}(s)$.[11]

A set of compatible set attributes is said to be *complete* if the join of their partitions is discrete, i.e., the blocks have cardinality 1. A *Complete Set of Compatible Attributes* or *CSCA* characterizes the singletons $\{u\} \subseteq U$ by the ordered n-tuple $(r, ..., s)$ of attribute values.

All this machinery of set partitions can be lifted or transported to vector spaces to give the mathematical machinery of QM.[12]

5 Lifting partition math from sets to vector spaces

5.1 The basis principle

There is a natural part-of-the-folklore bridge or ladder for lifting set concepts to vector-space concepts. The basic idea is that a vector $v = \sum_i \alpha_i b_i$, represented in terms of a set $\{b_i\}$ of basis vectors, is a *K-valued set* where each element b_i in the basis set takes a value c_i in the base field K. Given a set concept, the *basis principle* is that one can generate the corresponding vector-space concept by applying the set concept to a set that is a basis and then seeing what it generates. Starting with the set concept of cardinality, one arrives at the corresponding vector-space concept by applying the set concept to a basis set to arrive at the cardinality of the basis set. After checking that all bases have the same cardinality, this yields the vector-space notion of *dimension.* Thus the cardinality of a set *lifts* not to the cardinality of a vector space but to its dimension.

Some of the lifting is accomplished by the free vector space functor from the category of sets to the category of vector spaces over a given field K. A set U is carried by this functor to the vector space K^U spanned by the Kronecker delta basis $\{\delta_u : U \to K\}_{u \in U}$ where $\delta_u(u') = 0$ for $u' \neq u$ and $\delta_u(u) = 1$. A set U of a certain cardinality thus generates a vector space K^U of the same dimension.

[10] Technically, a "distinction" of a partition $\pi = \{B\}$ on U is an ordered pair (u_i, u_j) of elements of U in different blocks of the partition.

[11] These set-based notions of "eigenvalue" and "eigenvector" will be later developed in QM/sets.

[12] In QM, the extension of concepts on finite dimensional Hilbert space to infinite dimensional ones is well-known. Since our expository purpose is conceptual rather than mathematical, we will stick to finite dimensional spaces.

5.2 What is a vector space partition?

A partition $\pi = \{B\}$ on a set U is a set of subsets whose direct sum (i.e., disjoint union) is the whole set, i.e., a direct sum decomposition of the set. The corresponding vector space concept is a set of subspaces of a vector space V whose direct sum is the vector space, i.e., a *direct sum decomposition* of the vector space. In terms of the basis principle, we could apply the set partition $\pi = \{B\}$ of a set U to a basis set $\{b_u\}_{u \in U}$ of V, then each block B generates a subspace $V_B \subseteq V$ and the set of subspaces $\{V_B\}_{B \in \pi}$ is a direct sum decomposition of the vector space V. Thus the lift of the concept of a set partition is a direct sum decomposition of a vector space. *Nota bene*, this notion of a vector space partition (direct sum decomposition) is *not* a set partition of a vector space that is compatible with the vector space operations, i.e., a quotient space V/W as would be defined by each subspace $W \subseteq V$ with the equivalence relation $v \sim v'$ if $v - v' \in W$ (see [18]). While a partition on a set is essentially the same as a quotient set (or equivalence relation on the set), the vector-space lift of a set partition is *not* a quotient vector space but a direct sum decomposition of a vector space.

This lifting of set concepts to vector space concepts is not particularly new; it is part of the mathematical folklore. Hermann Weyl outlined the lifting sets to vector spaces program by first considering an attribute on a set, which defined the set partition or "grating" [47, p. 255] of elements with the same attribute-value. Then he moved to the quantum case where the set or "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [47, p. 256].[13] The appropriate notion of a partition or "grating" is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\overrightarrow{x}$ splits into r component vectors lying in the several subspaces" [47, p. 256], i.e., a direct sum decomposition of the space, where the subspaces are the eigenspaces of an observable operator.

5.3 What is a vector space attribute?

A set attribute is a function $f : U \to \mathbb{R}$ (where the set of values is taken as the reals). The inverse-image $f^{-1}(r) \subseteq U$ of each value $f(u) = r$ is a subset where the attribute has the same value, and those subsets form a set partition. Given a basis set $\{b_u\}_{u \in U}$ of a vector space V over a field K, we can apply a set attribute $f : \{b_u\}_{u \in U} \to K$ to the basis set and see what it generates. One possibility is to linearly extend the function $f^*(b_u) = f(b_u)$ to the whole space to obtain a linear functional $f^* : V \to K$. But a linear functional defines a quotient space $V/f^{*-1}(0)$, not a vector space partition (direct sum decomposition).

The same information $f : \{b_u\}_{u \in U} \to K$ *also* defines $\hat{f}(b_u) = f(b_u) b_u$ which linearly extends to a *linear operator* $\hat{f} : V \to V$. The given basis vectors $\{b_u\}$ are eigenvectors of the operator $\hat{f}$ with the eigenvalues $f(b_u)$, and the eigenspaces are the subspaces where the operator has the same eigenvalue. The eigenvectors span the whole space so we see that the lift of a set attribute, which defines a set partition, is a vector space linear operator whose eigenspaces are a vector space partition (i.e., direct sum decomposition) of the whole space, i.e., a *diagonalizable linear operator*.[14]

6 Whence vector-space partitions?

6.1 Vector-space partitions from vector-space attributes

Given a diagonalizable linear operator $L : V \to V$, where V is a finite-dimensional vector space over a field K and where $\lambda_1, ..., \lambda_k$ are the distinct eigenvalues, then there are projection operators P_i for $i = 1, ..., k$ such that:

[13] Note the lift from sets to vector spaces using the basis principle where the cardinality n becomes dimension n.

[14] The natural connection between a function $f : U \to \mathbb{R}$ and an operator will be further explained in the later treatment of QM/sets.

1. $L = \sum_{i=1}^{k} \lambda_i P_i$;

2. $I = \sum_{i=1}^{k} P_i$;

3. $P_i P_j = 0$ for $i \neq j$; and

4. the range of P_i is the eigenspace V_i for the eigenvalue λ_i for $i = 1, ..., k$. [23, Theorem 8, p. 172]

What is the vector space partition *canonically* defined by a diagonalizable linear operator? Any basis of eigenvectors could be seen as defining a direct sum of the one-dimensional subspaces spanned by those eigenvectors. But those subspaces are far from unique. But if we group together all the eigenvectors with the same eigenvalue (i.e., use the top-down method to define a vector-space partition), then they span the eigenspaces. It is the set of eigenspaces $\{V_i\}$ that gives the unique canonical direct-sum decomposition or vector-space partition defined by a (diagonalizable) linear operator. This standard linear algebra result holds for any base field, but for QM, the base field is the complex numbers $\mathbb{C}$. In order for the eigenvalues to always be real, a diagonalizable linear operator is required to be *Hermitian* (or *self-adjoint*, i.e., equal to its conjugate transpose or adjoint). Thus the direct sum decomposition of a vector space by the eigenspaces of a diagonalizable linear operator is the vector-space version of the top-down method of defining partitions. What is the vector-space version of the bottom-up method of defining partitions?

6.2 Vector-space partitions from vector-space representations of groups

A *vector-space representation* of an abstract group G is a group homomorphism $T = \{T_g\} : G \to GL(V)$ where $GL(V)$ is the group of invertible linear transformations $V \to V$ of a vector space V over the complex numbers. Here again, the idea is to define a (vector-space) partition by a (linear) group of transformations $T_g : V \to V$ that map elements $v \in V$ to similar or symmetric elements $T_g(v)$. A subspace $W \subseteq V$ is *invariant* if $T_g(W) \subseteq W$ for all $g \in G$. And again, it is the minimal invariant subspaces, the *irreducible subspaces*, that are of interest. The irreducible subspaces $\{W_\alpha\}$ are the carriers for the *irreducible representations* $T \restriction : W_\alpha \to W_\alpha$ or *irreps*. And the representation space V is a direct sum of some set of irreducible subspaces $V = \sum_{i=1}^{l} \oplus W_i$ so the vector-space representation of a group defines a vector-space partition of the space. But these vector-space partitions are not unique and are thus not canonically defined by the representation. George Mackey poses this problem of finding the bottom-up analogue to the top-down determination of the unique direct sum decomposition using the eigenspaces of an operator.

> Finding such a decomposition [of irreps] is an exact analogue of finding a basis of eigenvectors of a single operator. In neither case is the decomposition unique. However, in the operator case the eigenvalues and the multiplicities of occurrence are uniquely determined. Moreover the linear span of these basis vectors with a common eigenvalue is just the total eigenspace for that eigenvalue and is uniquely determined. The decomposition as a direct sum of eigenspaces is unique. [32, p. 244]

Hence the problem in this bottom-up approach is "finding an analogue for equality of eigenvalues" [32, p. 244] to group the irreps together.

Suppose T is a representation of G acting on a space V and T' is a representation of the same G acting on a space V'. Then a linear map $\phi : V \to V'$ is said *morphism of representations* or *intertwining map* if for all $g \in G$ and all $v \in V$:

$$\phi(T_g(v)) = T'_g(\phi(v)), \text{ i.e.,}$$

$$\begin{array}{ccc} V & \xrightarrow{T_g} & V \\ \phi\downarrow & & \downarrow\phi \\ V' & \xrightarrow{T'_g} & V' \end{array}$$

commutes.

If ϕ is also invertible, then ϕ is said to be an *isomorphism* of representations, and the representations are said to be *isomorphic* or *equivalent*.

The remarkable fact is that each group has a fixed set of inequivalent irreps, so the distinct irreps are a characteristic of the group itself, not of a particular representation.

The uniqueness and canonical nature of the partition obtained in the operator case by equality of eigenvalues is now obtained using equivalence of irreps and their underlying irreducible subspaces. All the irreducible subspaces W_i for irreps equivalent to an irrep L in any such direct sum $V = \sum_{i=1}^{l} \oplus W_i$ are summed together to obtain the invariant carrier W_L for a primary representation–where a representation is *primary* if all its irreducible subrepresentations are equivalent and the underlying carrier space is also called *primary*. The decomposition of V as the direct sum $\sum_L \oplus W_L$ of the invariant primary subspaces for the primary representations is unique. "It is the invariant subspaces $[W_L]$ which are the analogues of the eigenspaces of a single operator." [32, p. 244] In terms of representations rather than their carrier subspaces, it is the unique "*canonical decomposition into primary representations*." [32, p. 244]

Thus we have the top-down construction of the vector space partition $V = \sum \oplus V_i$ of eigenspaces V_i given by an operator (or vector-space attribute) and the bottom-up construction of the vector-space partition $V = \sum \oplus W_L$ of the carriers W_L for primary representations given by a vector-space representation of a symmetry group.

The following table brings out the analogies between the top-down and bottom-up determination of vector-space partitions.

Figure 7: Top-down and bottom-up determinations of vector-space partitions

To represent indefiniteness, we first need to specify the "universe" of fully definite eigen-alternatives, and then indefiniteness can be described by superposing the "potential" eigen-alternatives. In the vector-space case, the eigen-alternatives top-down determined by an operator are the eigenvectors and the eigen-alternatives bottom-up determined by a representation of a symmetry group are the minimal invariant subspaces that are the carriers for the irreducible representations of the symmetry group. It is this role of determining the eigen-alternatives that are superposed to represent indefiniteness which gives the irreps of symmetry groups a more fundamental role in QM than in classical physics and engineering.

> It can in fact be shown that every state of any quantum mechanical system, no matter what type of interactions are present, can be considered as a superposition of states of elementary systems. The elementary systems correspond mathematically to irreducible representations of the Lorentz group and as such can be enumerated. [49, p. 8]

For state-dependent (or extrinsic) attributes of a quantum particle like the linear momentum or angular momentum, the fully definite eigenstates are determined by the irreducible representations of the linear-translation or rotational-translation symmetry groups respectively. For the state-independent (or intrinsic) attributes of quantum particles, like having a mass, charge, and spin, they are determined in particle physics by the irreducible representations of the appropriate symmetry groups.[15]

6.3 Vector-space partitions from other vector-space partitions

The set notion of compatibility lifts to vector spaces, via the basis principle, by defining two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ on V as being *compatible* if there is a basis set for V so that the two vector space partitions are generated by two set partitions of that common or simultaneous basis set.

If two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ are compatible, then their *vector space join* $\omega \vee \xi$ is defined as the vector space partition whose subspaces are the non-zero intersections $W_\lambda \cap X_\mu$. And by the definition of compatibility, we could also generate the subspaces of the join $\omega \vee \xi$ by the blocks in the set join of the two set partitions of the common basis set.

Since real-valued set attributes lift to Hermitian linear operators, the notion of compatible set attributes just defined would lift to two linear operators being *compatible* if their eigenspace partitions are compatible. It is a standard fact of linear algebra [23, p. 177] that two diagonalizable linear operators $L, M : V \to V$ (on a finite dimensional space V) are compatible in the sense of having a basis of simultaneous eigenvectors if and only if they commute, $LM = ML$. Hence the *commutativity* of linear operators is the lift of the compatibility (i.e., defined on the same set) of set attributes. That explains the importance of the notion of commutativity in QM and that is why the repeated compatible measurements, described mathematically as the join operation, requires commutativity. The join of two operator-determined eigenspace partitions is defined iff the operators commute. As Weyl put it: "Thus combination [DE: join] of two gratings [eigenspace partitions of two operators] presupposes commutability...". [47, p. 257]

Two commuting Hermitian linear operators L and M have compatible eigenspace partitions $W_L = \{W_\lambda\}$ (for the eigenvalues λ of L) and $W_M = \{W_\mu\}$ (for the eigenvalues μ of M). The blocks in the join $W_L \vee W_M$ of the two compatible eigenspace partitions are the non-zero subspaces $\{W_\lambda \cap W_\mu\}$ which can be characterized by the ordered pairs of eigenvalues (λ, μ). The nonzero vectors $v \in W_\lambda \cap W_\mu$ are *simultaneous eigenvectors* for the two commuting operators, and there is a basis for the space consisting of simultaneous eigenvectors.[16]

A set of commuting linear operators is said to be *complete* if the join of their eigenspace partitions is nondegenerate, i.e., the blocks have dimension 1. The join operation gives the results of compatible measurements so the join of a complete set of compatible vector space attributes (i.e., commuting Hermitian operators) gives the possible results of a non-degenerate measurement. The eigenvectors that generate those one-dimensional blocks of the join are characterized by the ordered n-tuples $(\lambda, ..., \mu)$ of eigenvalues so the eigenvectors are usually denoted as the eigenkets $|\lambda, ..., \mu\rangle$ in the Dirac notation. These *Complete Sets of Commuting Operators* are Dirac's CSCOs [9] (which are the vector space version of our previous CSCAs).[17]

[15] The classic paper in this group-theoretic treatment of particles is Wigner [48]. For recent overviews, see the group-theoretical definition of particles in Falkenburg [15] or Roberts [37].

[16] One must be careful not to assume that the simultaneous eigenvectors are the eigenvectors for the operator $LM = ML$ due to the problem of degeneracy.

[17] In both the cases of set-partitions and vector-space partitions defined in the top-down manner by attributes or

Since the eigen-alternatives determined by an operator, i.e., eigenvectors, can be obtained by the complete partition joins defined by a CSCO, one might ask if the eigen-alternatives determined by a group representation, i.e., the irreps and their irreducible carrier spaces, could also be obtained by the partition joins defined by some CSCO. Jin-Quan Chen, Fan Wang, and their colleagues in the Nanjing School have developed a little-known CSCO method to systematically find the irreducible basis vectors for the irreducible spaces that works not only for all representations of finite groups but for all compact Lie groups as needed in QM ([6], [7]). "[T]he foundation of the new approach is precisely the theory of the complete set of commuting operators (CSCO) initiated by Dirac..." [7, p. 2] Thus the vector-space partition joins of the CSCO method extends also to all compact group representations to characterize the maximally definite eigen-alternatives.

The partitional mathematics for sets and vector spaces is summarized in the following table.

Figure 8: Summary of partition concepts for sets and vector spaces

Indefiniteness is mathematically treated using partitions. We have now shown how partitions are defined by the top-down and bottom-up methods in both the cases of sets and complex vector spaces, and we noted that the complex vector space notions (e.g., eigenvectors and irreps) are central in the mathematics of full QM. It remains to show that there is also a pedagogical or "toy" model of QM over sets, QM/sets, that will illuminate the mathematics of QM in the simple setting of sets.

operators, we have emphasized "making distinctions" by the join operation between compatible partitions. In the case of set or vector-space partitions defined in the bottom-up manner by symmetry groups, the way to "make distinctions" is *breaking symmetry* by moving to a subgroup of the symmetry group. This way of "making distinctions" is important in particle physics and "Big Bang" cosmology but we will not further consider it here.

7 Quantum mechanics over sets or QM/sets

7.1 Objective indefiniteness in probability theory

The quantum probability calculus of QM/sets is the classical Laplace-Boole finite probability theory–albeit in a non-commutative version. But the interpretation is different; the "events" are not subjective states of knowledge but are objective states of being (about which there may or may not be observers with subjective states of knowledge).

Since our purpose is conceptual rather than mathematical, we will stick to the simplest case of Laplace-Boole ([29], [2]) finite probability theory with a finite sample space $U = \{u_1, ..., u_n\}$ of n equiprobable outcomes and to finite dimensional QM.[18] In the usual terms, the *events* are the subsets $S \subseteq U$, and the *probability* of an event S occurring in a trial is the ratio of the cardinalities: $\Pr(S) = \frac{|S|}{|U|}$. Given that a conditioning event $S \subseteq U$ occurs, the *conditional probability* that $T \subseteq U$ occurs is: $\Pr(T|S) = \frac{\Pr(T \cap S)}{\Pr(S)} = \frac{|T \cap S|}{|S|}$. The ordinary probability $\Pr(T)$ of an event T can be taken as the conditional probability with U as the conditioning event so all probabilities can be seen as conditional probabilities. Given a (real-valued) random variable, i.e., a *numerical attribute* $f : U \to \mathbb{R}$ on the elements of U, the *probability of observing a value r given an event S* is the conditional probability of the event $f^{-1}(r)$ given S:

$$\Pr(r|S) = \frac{|f^{-1}(r) \cap S|}{|S|}.$$

That is all the probability theory we will need here.

As noted, that formal probability theory is now to be interpreted in the different terms of objective states of being, not subjective states of knowledge. The elements u of the "sample space" U considered as the singletons $\{u\}$ are the definite states, the *eigenstates* of definiteness.[19] Collecting together a number of eigenstates into a multiple-element subset $S \subseteq U$ is the *superposition* of those definite eigenstates $\{u\} \subseteq S$. Thus a multiple-element subset or "event" S is interpreted as an objective state that is *objectively indefinite* between the definite eigenstates $\{u\} \subseteq S$. In short, we are using a different interpretation of a "set." A set $\{a, b\}$ is not to be thought of as a collection of two distinct things $\{a\}$ and $\{b\}$, but as a single indefinite thing that is characterized as being indefinite between $\{a\}$ and $\{b\}$.

Instead of being given the epistemological state of the conditioning event S, we are always given an objective state S which could be U.[20] Then the conditional probability $\Pr(T|S)$ is interpreted as the probability that S will reduce or "collapse" to the more definite objective state $T \cap S \subseteq S$ when an experiment is made that is a "measurement" of a numerical attribute on U.

In the usual presentation of probability theory, the numerical attribute associated with an event T may be left implicit but it can be taken as the characteristic function $\chi_T : U \to \{0, 1\} \subseteq \mathbb{R}$ so that the conditional probability $\Pr(T|S) = \Pr(T \cap S|S)$ is the probability that the measurement of the attribute χ_T in the state S returns the value of 1, i.e.,

$$\Pr(1|S) = \frac{|\chi_T^{-1}(1) \cap S|}{|S|} = \frac{|T \cap S|}{|S|} = \Pr(T \cap S|S) = \Pr(T|S).$$

In this manner, the "trial" or "experiment" in the usual epistemological interpretation of finite probability theory can always be seen as a "measurement" of a numerical attribute that "reduces" or "collapses" the state of knowledge from S to $T \cap S$. In the objective indefiniteness interpretation

[18] The mathematics can be generalized to the case where each point u_i in the sample space has a probability p_i but the simpler case of equiprobable points serves our conceptual purposes.

[19] However, when we later consider the singletons of the U-elements as just one basis set among many in the vector space $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$, then we will see that the $\{u\}$ are definite for some attributes but may be completely indefinite for other attributes.

[20] The empty subset $\emptyset$ is not considered as an objective state so $S \neq \emptyset$.

used here, a state reduction is also made but it is an objective state rather than just a state of knowledge that is reduced or "collapsed" when a measurement-experiment is performed.

Our next task is to show how the mathematics of the classical finite probability theory can be recast as a quantum probability calculus for the pedagogical model of QM/sets where the objective states are subsets $S \subseteq U$ seen as vectors in a vector space over $\mathbb{Z}_2$. We will throughout emphasize the analogies with the full QM mathematics in vector spaces over $\mathbb{C}$ with inner products.

7.2 Previous attempts to develop QM over $\mathbb{Z}_2$

Quantum mechanics over sets, QM/sets, is a bare-bones "logical" (e.g., non-physical[21]) version of QM with appropriate set-versions of spectral decomposition, the Dirac brackets, ket-bra resolution, the norm, observable-attributes, and the Born rule all in the simple classical setting of sets, but that nevertheless provide models of characteristically quantum results (e.g., a QM/sets version of the double-slit experiment).

There have been at least four previous attempts at developing a version of QM over sets, i.e., where the base field of $\mathbb{C}$ is replaced by $\mathbb{Z}_2$ ([38], [21], [44], and [1]). All these attempts use the aspect of full QM that the brackets (and observables) take their values in the base field. When the base field is $\mathbb{Z}_2$, then the models do "not make use of the idea of probability"[38, p. 919] and have instead only a modal interpretation (1 = possibility and 0 = impossibility).

The model of QM over sets developed here is based on a different understanding of the relation between the pedagogical model and full QM. The relation between QM/sets and full QM is not that of two models of the same set of abstract axioms (e.g., as in [1]) but as a *progression* of internalizing features as the base field is increased from $\mathbb{Z}_2$ to $\mathbb{C}$. QM/sets can then perfectly well have the brackets and observables that take external values *outside* the base field of $\mathbb{Z}_2$ (e.g., use real-valued observables = real-valued random variables in classical finite probability theory) and even define a more primitive version of "eigenvectors" and "eigenvalues" that are not (in general) the eigenvectors and eigenvalues of linear operators on the vector space over $\mathbb{Z}_2$. The increased power of $\mathbb{C}$ (e.g., algebraic completeness) then allows the "proto-eigenvectors" and "proto-eigenvalues" of QM/sets to be "internalized" as true eigenvectors and eigenvalues of (Hermitian) linear operators on vector spaces over $\mathbb{C}$ and the brackets can then also be "internalized" as a bilinear inner product taking values in the base field $\mathbb{C}$. Hence under this approach (and in contrast to the four previous approaches), the "taking values in the base field" is seen *only* as an aspect of full QM over $\mathbb{C}$ and not as a necessary aspect of a pedagogical proto-QM model such as QM/sets with the base field of $\mathbb{Z}_2$.

7.3 Vector spaces over 2

To show how classical Laplace-Boole finite probability theory can be recast as a quantum probability calculus, we use finite dimensional vector spaces over $\mathbb{Z}_2$. The power set $\wp(U)$ of $U = \{u_1, ..., u_n\}$ is a vector space over $\mathbb{Z}_2 = \{0, 1\}$, isomorphic to $\mathbb{Z}_2^n$, where the vector addition $S + T$ is the *symmetric difference* (or inequivalence) of subsets. That is, for $S, T \subseteq U$,

$$S + T = (S - T) \cup (T - S) = S \cup T - S \cap T.$$

The *U-basis* in $\wp(U)$ is the set of singletons $\{u_1\}, \{u_2\}, ..., \{u_n\}$. A vector $S \in \wp(U)$ is specified in the U-basis as $S = \sum_{u \in S} \{u\}$ and it is characterized by its $\mathbb{Z}_2$-valued characteristic function $\chi_S : U \to \mathbb{Z}_2 \subseteq \mathbb{R}$ of coefficients since $S = \sum_{u \in U} \chi_S(u) \{u\}$. Similarly, a vector v in $\mathbb{C}^n$ is specified in terms of an orthonormal basis $\{|v_i\rangle\}$ as $v = \sum_i c_i |v_i\rangle$ and is characterized by a $\mathbb{C}$-valued function $\langle \_|v\rangle : \{v_i\} \to \mathbb{C}$ assigning a complex amplitude $\langle v_i|v\rangle = c_i$ to each basis vector $|v_i\rangle$.

[21] In full QM, the DeBroglie relations connect mathematical notions such as frequency and wave-length to physical notions such as energy and momentum. QM/sets is "non-physical" in the sense that it is a sets-version of the pure mathematical framework of (finite-dimensional) QM without those direct physical connections.

One of the key pieces of mathematical machinery in QM, namely the inner product, does not exist in vector spaces over finite fields but brackets and a norm can still be defined to play a similar role in the probability calculus of QM/sets.

Seeing $\wp(U)$ as the abstract vector space $\mathbb{Z}_2^n$ allows different bases in which the vectors can be expressed (as well as the basis-free notion of a vector as a "ket"). Hence the quantum probability calculus developed here can be seen as a "non-commutative" generalization of the classical Laplace-Boole finite probability theory where a different basis corresponds to a different equicardinal sample space $U' = \{u'_1, ..., u'_n\}$.

Consider the simple case of $U = \{a, b, c\}$ where the U-basis is $\{a\}$, $\{b\}$, and $\{c\}$. The three subsets $\{a, b\}$, $\{b, c\}$, and $\{a, b, c\}$ also form a basis since:

$\{b, c\} + \{a, b, c\} = \{a\}$;

$\{b, c\} + \{a, b\} + \{a, b, c\} = \{b\}$; and

$\{a, b\} + \{a, b, c\} = \{c\}$.

These new basis vectors could be considered as the basis-singletons in another equicardinal universe $U' = \{a', b', c'\}$ where $\{a'\}$, $\{b'\}$, and $\{c'\}$ refer to the same abstract vector or *ket* as $\{a, b\}$, $\{b, c\}$, and $\{a, b, c\}$ respectively.

In the following *ket table*, each row is an abstract vector of $\mathbb{Z}_2^3$ expressed in the U-basis, the U'-basis, and a U''-basis.

$U = \{a, b, c\}$	$U' = \{a', b', c'\}$	$U'' = \{a'', b'', c''\}$
$\{a, b, c\}$	$\{c'\}$	$\{a'', b'', c''\}$
$\{a, b\}$	$\{a'\}$	$\{b''\}$
$\{b, c\}$	$\{b'\}$	$\{b'', c''\}$
$\{a, c\}$	$\{a', b'\}$	$\{c''\}$
$\{a\}$	$\{b', c'\}$	$\{a''\}$
$\{b\}$	$\{a', b', c'\}$	$\{a'', b''\}$
$\{c\}$	$\{a', c'\}$	$\{a'', c''\}$
$\emptyset$	$\emptyset$	$\emptyset$

Vector space isomorphism: $\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U') \cong \wp(U'')$ where row = ket.

In the Dirac notation [9], the *ket* $|\{a, c\}\rangle$ represents the abstract vector that is represented in the U-basis as $\{a, c\}$. A row of the ket table gives the different representations of the *same* ket in the different bases, e.g., $|\{a, c\}\rangle = |\{a', b'\}\rangle = |\{c''\}\rangle$.

7.4 The brackets

In a Hilbert space, the inner product is used to define the brackets $\langle v_i | v\rangle$ and the norm $|v| = \sqrt{\langle v|v\rangle}$. In a vector space over $\mathbb{Z}_2$, the Dirac notation can still be used to define the brackets and norm even though there is no inner product. For a singleton basis vector $\{u\} \subseteq U$, the *bra* $\langle \{u\}|_U : \wp(U) \to \mathbb{R}$ is defined by the *bracket*:

$$\langle \{u\} |_U S\rangle = \begin{cases} 1 \text{ if } u \in S \\ 0 \text{ if } u \notin S \end{cases} = |\{u\} \cap S| = \chi_S(u).$$

Note that the bra and the bracket is defined in terms of the U-basis and that is indicated by the U-subscript on the bra portion of the bracket. Then for $u_i, u_j \in U$, $\langle \{u_i\} |_U \{u_j\}\rangle = \chi_{\{u_j\}}(u_i) = \chi_{\{u_i\}}(u_j) = \delta_{ij}$ (the Kronecker delta function) which is the QM/sets-version of $\langle v_i|v_j\rangle = \delta_{ij}$ for an orthonormal basis $\{|v_i\rangle\}$ of $\mathbb{C}^n$. The bracket linearly extends *in the reals* (not in the base field $\mathbb{Z}_2$) to any two vectors $T, S \in \wp(U)$:[22]

[22]Here $\langle T|_U S\rangle = |T \cap S|$ takes values outside the base field of $\mathbb{Z}_2$ just like, say, the Hamming distance function $d_H(T, S) = |T + S|$ on vector spaces over $\mathbb{Z}_2$ in coding theory. [34] Thus the bra $\langle S|_U$ is not to be confused with

$$\langle T|_U S\rangle = |T \cap S|.$$

This is the QM/sets-version of the Dirac brackets in the mathematics of QM.

For more motivation, consider an orthonormal basis set $\{|v_i\rangle\}_{i=1,...,n}$ in an n-dimensional Hilbert space V and the association $\{u_i\} \leftrightarrow |v_i\rangle$ for $i = 1, ..., n$. Given two subsets $T, S \subseteq U$, $T = \sum_{u_i \in T} \{u_i\}$ corresponds to the unnormalized $\psi_T = \sum_{u_i \in T} |v_i\rangle$ and similarly for ψ_S. Then their inner product (defined using the $\{|v_i\rangle\}_{i=1,...,n}$ basis) in V is $\langle \psi_T | \psi_S \rangle = |T \cap S| = \langle T|_U S\rangle$. In both cases, the bracket gives a measure of the *overlap* or indistinctness of the two vectors.[23]

7.5 Ket-bra resolution

The *ket-bra* $|\{u\}\rangle \langle\{u\}|_U$ is defined as the one-dimensional projection operator:

$$|\{u\}\rangle \langle\{u\}|_U = \{u\} \cap () : \wp(U) \to \wp(U)$$

and the *ket-bra identity* holds as usual:

$$\sum_{u \in U} |\{u\}\rangle \langle\{u\}|_U = \sum_{u \in U} (\{u\} \cap ()) = I : \wp(U) \to \wp(U)$$

where the summation is the symmetric difference of sets in $\wp(U)$ and I is the identity map [as a linear operator on $\wp(U)$]. The overlap $\langle T|_U S\rangle$ can be resolved using the ket-bra identity in the same basis:

$$\langle T|_U S\rangle = \sum_u \langle T|_U \{u\}\rangle \langle\{u\}|_U S\rangle$$

where the sum of real-valued brackets is over the reals.
Similarly a ket $|S\rangle$ for $S \subseteq U$ can be resolved in the U-basis;

$$|S\rangle = \sum_{u \in U} |\{u\}\rangle \langle\{u\}|_U S\rangle = \sum_{u \in U} \langle\{u\}|_U S\rangle |\{u\}\rangle = \sum_{u \in U} |\{u\} \cap S| |\{u\}\rangle$$

where a subset $S \subseteq U$ is just expressed as the sum of the singleton subsets $\{u\} \subseteq S$. That is *ket-bra resolution* in QM/sets. The ket $|S\rangle$ is the same as the ket $|S'\rangle$ for some subset $S' \subseteq U'$ in another U'-basis, but when the bra $\langle\{u\}|_U$ is applied to the ket $|S\rangle = |S'\rangle$, then it is the subset $S \subseteq U$, not $S' \subseteq U'$, that comes outside the ket symbol $|\ \rangle$ in $\langle\{u\}|_U S\rangle = |\{u\} \cap S|$.[24]

7.6 The norm

The *U-norm* $\|S\|_U : \wp(U) \to \mathbb{R}$ is defined, as usual, as the square root of the bracket:

$$\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S \cap S|} = \sqrt{|S|}$$

for $S \in \wp(U)$ which is the QM/sets-version of the norm $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$ in ordinary QM. Note that a ket has to be expressed in the U-basis to apply the U-norm definition so, for example, $\|\{a'\}\|_U = \sqrt{2}$ since $|\{a'\}\rangle = |\{a, b\}\rangle$.

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the dual functional $\chi_S : \wp(U) \to \mathbb{Z}_2$ that does take values in the base field. The brackets taking values in the base field is a consequence of the base field being strengthened to $\mathbb{C}$. It is not a necessary feature of a quantum probability calculus as we see in QM/sets.

[23]Indeed in QM/sets, the brackets $\langle T|_U S\rangle = |T \cap S|$ for $T, T', S \subseteq U$ should be thought of *only* as a measure of the overlap since they are not even linear, e.g., $\langle T + T'|_U S\rangle \neq \langle T|_U S\rangle + \langle T'|_U S\rangle$ whenever $T \cap T' \neq \emptyset$. Only as the base field $\mathbb{Z}_2$ is increased to $\mathbb{C}$ (or $\mathbb{R}$) do the brackets 'fall into place' as a bilinear inner product. QM/sets is not 'supposed' to have completely the same mathematical structure as QM only with $\mathbb{Z}_2$ replacing $\mathbb{C}$. QM/sets is a proto-QM where things only 'fall into place' and are 'internalized' as the transition is made from $\mathbb{Z}_2$ to $\mathbb{C}$ as the base field.

[24]The term "$\{u\} \cap S'$" is not even defined since it is the intersection of subsets $\{u\} \subseteq U$ and $S' \subseteq U'$ of two different universe sets U and U'.

7.7 Numerical attributes and linear operators

In classical physics, the observables are numerical attributes, e.g., the assignment of a position and momentum to particles in phase space. One of the differences between classical and quantum physics is the replacement of these observable numerical attributes by linear operators associated with the observables where the values of the observables appear as eigenvalues of the operators. But this difference may be smaller than it would seem at first since a numerical attribute $f : U \to \mathbb{R}$ can be recast into an operator-like format in QM/sets, and there is even a QM/sets-analogue of spectral decomposition.

An observable, i.e., a Hermitian operator, on a Hilbert space V has a home basis set of orthonormal eigenvectors. In a similar manner, a real-valued attribute $f : U \to \mathbb{R}$ defined on U has the U-basis as its "home basis set." The connection between the numerical attributes $f : U \to \mathbb{R}$ of QM/sets and the Hermitian operators of full QM can be established by seeing the function f as being *like* an "operator" $f \upharpoonright ()$ on $\wp(U)$ [where $f \upharpoonright S$ is the *restriction* of f to $S \in \wp(U)$] in defining an "eigenvalue equation." For any subset $S \in \wp(U)$, consider the definition:

The *eigenvalue equation*: $f \upharpoonright S = rS$ holds iff f is constant on the subset S with the value r.

This is the QM/sets-version or proto-version of an eigenvalue equation for general numerical attributes $f : U \to \mathbb{R}$. Whenever S satisfies $f \upharpoonright S = rS$ (i.e., S is a constant or level set of f) for some r, then S is said to be an *eigenvector* of the numerical attribute $f : U \to \mathbb{R}$ in the vector space $\wp(U)$, and $r \in \mathbb{R}$ is the associated *eigenvalue*.[25] Each eigenvalue r determines "as usual" (i.e., like in full QM) an *eigenspace* $\wp\left(f^{-1}(r)\right)$ of its eigenvectors which is a subspace of the vector space $\wp(U)$. The whole space $\wp(U)$ can be expressed as usual as the direct sum of the eigenspaces: $\wp(U) = \bigoplus_{r \in f(U)} \wp\left(f^{-1}(r)\right)$. Note that in QM/sets, an attribute f defines both the set partition $\left\{f^{-1}\right\} = \left\{f^{-1}(r\right\}_{r \in f(U)}$ and the vector-space partition or direct sum decomposition of the eigenspaces $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r \in f(U)}$. The set partition $\left\{f^{-1}\right\}$ chops up (via the join operation) an arbitrary subset or vector $S \in \wp(U)$ into the parts $f^{-1}(r) \cap S$ which are the images in the eigenspaces $\wp\left(f^{-1}(r)\right)$ of the projection operators $f^{-1}(r) \cap () : \wp(U) \to \wp(U)$. This is all analogous to full QM where, as we will later see, the join operation of applying $\left\{f^{-1}\right\}$ to chop up the given state vector S to give the projected parts $f^{-1}(r) \cap S$ is the QM/sets version of *measurement*. Moreover, for distinct eigenvalues $r \neq r'$, any corresponding eigenvectors $S \in \wp\left(f^{-1}(r)\right)$ and $T \in \wp\left(f^{-1}(r')\right)$ are *orthogonal* in the sense that $\langle T|_U S\rangle = 0$. In general, for vectors $S, T \in \wp(U)$, orthogonality means zero overlap, i.e., disjointness.

The characteristic function $\chi_S : U \to \mathbb{R}$ for $S \subseteq U$ has the eigenvalues of 0 and 1 in the base field $\mathbb{Z}_2$ so it is a numerical attribute that *can* be "internalized" as a linear operator $S \cap () : \wp(U) \to \wp(U)$. Hence in this case, the "eigenvalue equation" $f \upharpoonright T = rT$ for $f = \chi_S$ becomes an actual eigenvalue equation $S \cap T = rT$ for the linear[26] operator $S \cap ()$ with the resulting eigenvalues of 1 and 0, and with the resulting eigenspaces $\wp(S)$ and $\wp(S^c)$ (where S^c is the complement of S) agreeing with those "eigenvalues" and "eigenspaces" defined above for an arbitrary numerical attribute $f : U \to \mathbb{R}$. The characteristic attributes $\chi_S : U \to \mathbb{R}$ are characterized by the property that their value-wise product, i.e., $(\chi_S \bullet \chi_S)(u) = \chi_S(u)\chi_S(u)$, is equal to the attribute value $\chi_S(u)$, and that is reflected in the idempotency of the corresponding operators:

$$\wp(U) \xrightarrow{S\cap()} \wp(U) \xrightarrow{S\cap()} \wp(U) = \wp(U) \xrightarrow{S\cap()} \wp(U).$$

[25]The context is sufficient to distinguish between these "proto-eigenvectors" of QM/sets and the eigenvectors in the usual sense in full QM.

[26]It should be noted that the projection operator $S \cap () : \wp(U) \to \wp(U)$ is not only idempotent but linear, i.e., $(S \cap T_1) + (S \cap T_2) = S \cap (T_1 + T_2)$. Indeed, this is the distributive law when $\wp(U)$ is interpreted as a Boolean ring with intersection as multiplication.

Thus the operators $S \cap ()$ corresponding to the characteristic attributes χ_S are *projection operators*. In order for general real-valued attributes to be internalized as linear operators, in the way that characteristic functions χ_S were internalized as projection operators $S \cap ()$, the base field would have to be strengthened to $\mathbb{C}$ and that would take us, *mutatis mutandis*, from the probability calculus of QM/sets to that of full QM.

The (maximal) eigenvectors $f^{-1}(r)$ for f, with r in the *image* or *spectrum* $f(U) \subseteq \mathbb{R}$, span the set U, i.e., $U = \sum_{r \in f(U)} f^{-1}(r)$. Hence the attribute $f : U \to \mathbb{R}$ has a spectral decomposition in terms of its (projection-defining) characteristic functions:

$$f = \sum_{r \in f(U)} r \chi_{f^{-1}(r)} : U \to \mathbb{R}$$
Spectral decomposition of set attribute $f : U \to \mathbb{R}$

which is the QM/sets-version of the spectral decomposition $L = \sum_\lambda \lambda P_\lambda$ of a Hermitian operator L in terms of the projection operators P_λ for its eigenvalues λ.

7.8 Completeness and orthogonality of projection operators

For any vector $S \in \wp(U)$, the operator $S \cap () : \wp(U) \to \wp(U)$ is the linear projection operator to the subspace $\wp(S) \subseteq \wp(U)$. The usual completeness and orthogonality conditions on projection operators P_λ to the eigenspaces of an observable-operator have QM/sets-versions for the projection operators $f^{-1}(r) \cap ()$ defined by numerical attributes $f : U \to \mathbb{R}$:

1. completeness: $\sum_\lambda P_\lambda = I : V \to V$ in QM has the QM/sets-version:

   $\sum_r f^{-1}(r) \cap () = I : \wp(U) \to \wp(U)$, and

2. orthogonality: for $\lambda \neq \mu$, $V \xrightarrow{P_\mu} V \xrightarrow{P_\lambda} V = V \xrightarrow{0} V$ (where 0 is the zero operator) has the QM/sets-version: for $r \neq r'$,

   $$\wp(U) \xrightarrow{f^{-1}(r') \cap ()} \wp(U) \xrightarrow{f^{-1}(r) \cap ()} \wp(U) = \wp(U) \xrightarrow{0} \wp(U).$$

Note that in spite of the lack of an inner product, the orthogonality of projection operators $S \cap ()$ is perfectly well-defined in QM/sets where it boils down to the disjointness of subsets, i.e., the cardinality of subsets' overlap (instead of their inner product) being 0.

7.9 The Born Rule for measurement in QM and QM/sets

An orthogonal decomposition of a finite set U is just a partition $\pi = \{B\}$ of U since the blocks $B, B', \ldots$ are orthogonal (i.e., disjoint) and their sum is U. Given such an orthogonal decomposition of U, we have the:

$$\|U\|_U^2 = \sum_{B \in \pi} \|B\|_U^2$$
Pythagorean Theorem
for orthogonal decompositions of sets.

An old question is: "why the squaring of amplitudes in the Born rule of QM?" A superposition state between certain definite orthogonal alternatives A and B, where the latter are represented by vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, is represented by the vector sum $\overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B}$. But what is the "strength," "intensity," or relative importance of the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ in the vector sum $\overrightarrow{C}$? That question requires a *scalar* measure of strength or intensity. The magnitude or "length" given by the norm $\|\|$ does not answer the question since $\left\|\overrightarrow{A}\right\| + \left\|\overrightarrow{B}\right\| \neq \left\|\overrightarrow{C}\right\|$. But the Pythagorean Theorem shows that the norm-squared gives the scalar measure of "intensity" that answers the question: $\left\|\overrightarrow{A}\right\|^2 + \left\|\overrightarrow{B}\right\|^2 = \left\|\overrightarrow{C}\right\|^2$

in vector spaces over $\mathbb{Z}_2$ or over $\mathbb{C}$. And when the superposition state is reduced by a measurement, then the *probability* that the indefinite state will reduce to one of the definite alternatives is given by that relative scalar measure of the eigen-alternative's "strength" or "intensity" in the indefinite state–and that is the Born Rule. In a slogan, Born is the off-spring of Pythagoras.

Given an orthogonal basis $\{|v_i\rangle\}$ in a finite dimensional Hilbert space and given the U-basis for the vector space $\wp(U)$, the corresponding Pythagorean results for the basis sets are:

$$\|\psi\|^2 = \sum_i \langle v_i|\psi\rangle^* \langle v_i|\psi\rangle \text{ and}$$
$$\|S\|_U^2 = \sum_{u\in U} \langle \{u\}\,|_U S\rangle \langle \{u\}\,|_U S\rangle$$

where the sum of the real-valued brackets is over the reals.

Given an observable-operator in QM and a numerical attribute in QM/sets, the corresponding Pythagorean Theorems for the complete sets of orthogonal projection operators are:

$$\|\psi\|^2 = \sum_\lambda \|P_\lambda(\psi)\|^2 \text{ and}$$
$$\|S\|_U^2 = \sum_r \left\|f^{-1}(r) \cap S\right\|_U^2 = \sum_r \left|f^{-1}(r) \cap S\right| = |S|.$$

Normalizing gives:

$$\sum_\lambda \frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2} = 1 \text{ and}$$
$$\sum_r \frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \sum_r \frac{\left|f^{-1}(r)\cap S\right|}{|S|} = 1$$

so the non-negative summands can be interpreted as probabilities–which is the Born rule in QM and in QM/sets.[27]

Here $\frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2}$ is the "mysterious" quantum probability of getting λ in an L-measurement of ψ, while $\frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ has the rather unmysterious interpretation in the pedagogical model, QM/sets, as the probability $\Pr(r|S)$ of the numerical attribute $f : U \to \mathbb{R}$ having the eigenvalue r when "measuring" $S \in \wp(U)$. Thus the QM/sets-version of the Born Rule is the perfectly ordinary Laplace-Boole rule for the conditional probability $\Pr(r|S) = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ that given $S \subseteq U$, a random variable $f : U \to \mathbb{R}$ takes the value r.

In QM/sets, when the indefinite state S is being "measured" using the observable f where the probability $\Pr(r|S)$ of getting the eigenvalue r is $\frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$, the "damned quantum jump" (Schrödinger) goes from S by the projection operator $f^{-1}(r) \cap ()$ to the projected resultant state $f^{-1}(r) \cap S$ which is in the eigenspace $\wp\left(f^{-1}(r)\right)$ for that eigenvalue r. The state resulting from the measurement represents a more-definite state $f^{-1}(r) \cap S$ that now has the definite f-value of r–so a second measurement would yield the same eigenvalue r with probability:

$$\Pr\left(r|f^{-1}(r) \cap S\right) = \frac{\left|f^{-1}(r)\cap\left[f^{-1}(r)\cap S\right]\right|}{\left|f^{-1}(r)\cap S\right|} = \frac{\left|f^{-1}(r)\cap S\right|}{\left|f^{-1}(r)\cap S\right|} = 1$$

and the same resulting vector $f^{-1}(r) \cap \left[f^{-1}(r) \cap S\right] = f^{-1}(r) \cap S$ using the idempotency of the projection operators.

Hence the treatment of measurement in QM/sets is *all* analogous to the treatment of measurement in full standard Dirac-von-Neumann QM.

[27] Note that there is no notion of a normalized vector in a vector space over $\mathbb{Z}_2$ (another consequence of the lack of an inner product). The normalization is, as it were, postponed to the probability algorithm which is computed in the reals. This "external" probability algorithm is "internalized" when $\mathbb{Z}_2$ is strengthened to $\mathbb{C}$ in going from QM/sets to full QM.

7.10 Summary of QM/sets and QM

The QM/set-versions of the corresponding QM notions are summarized in the following table for the finite U-basis of the $\mathbb{Z}_2$-vector space $\wp(U)$ and for an orthonormal basis $\{|v_i\rangle\}$ of a finite dimensional Hilbert space V.

QM/sets over $\mathbb{Z}_2$	Standard QM over $\mathbb{C}$
Projections: $S\cap() : \wp(U) \to \wp(U)$	$P : V \to V$ where $P^2 = P$
Spectral Decomposition.: $f = \sum_r r\chi_{f^{-1}(r)}$	$L = \sum_\lambda \lambda P_\lambda$
Completeness.: $\sum_r f^{-1}(r) \cap () = I$	$\sum_\lambda P_\lambda = I$
Orthog.: $r \neq r'$, $\left[f^{-1}(r)\cap()\right]\left[f^{-1}(r')\cap()\right] = \emptyset \cap ()$	$\lambda \neq \mu$, $P_\lambda P_\mu = 0$
Brackets: $\langle S\|_U T\rangle = \|S\cap T\|$ = overlap of $S, T \subseteq U$	$\langle\psi\|\varphi\rangle$ = overlap of ψ and φ
Ket-bra: $\sum_{u\in U} \|\{u\}\rangle\langle\{u\}\|_U = \sum_{u\in U}(\{u\}\cap()) = I$	$\sum_i \|v_i\rangle\langle v_i\| = I$
Resolution: $\langle S\|_U T\rangle = \sum_u \langle S\|_U\{u\}\rangle\langle\{u\}\|_U T\rangle$	$\langle\psi\|\varphi\rangle = \sum_i\langle\psi\|v_i\rangle\langle v_i\|\varphi\rangle$
Norm: $\\|S\\|_U = \sqrt{\langle S\|_U S\rangle} = \sqrt{\|S\|}$ where $S\subseteq U$	$\\|\psi\\| = \sqrt{\langle\psi\|\psi\rangle}$
Basis Pythagoras: $\\|S\\|_U^2 = \sum_{u\in U}\langle\{u\}\|_U S\rangle^2 = \|S\|$	$\\|\psi\\|^2 = \sum_i \langle v_i\|\psi\rangle^* \langle v_i\|\psi\rangle$
Normalized: $\sum_{u\in U}\frac{\langle\{u\}\|_U S\rangle^2}{\\|S\\|_U^2} = \sum_{u\in S}\frac{1}{\|S\|} = 1$	$\sum_i \frac{\langle v_i\|\psi\rangle^*\langle v_i\|\psi\rangle}{\\|\psi\\|^2} = 1$
Basis Born rule: $\Pr(\{u\}\|S) = \frac{\langle\{u\}\|_U S\rangle^2}{\\|S\\|_U^2}$	$\Pr(v_i\|\psi) = \frac{\langle v_i\|\psi\rangle^*\langle v_i\|\psi\rangle}{\\|\psi\\|^2}$
Attribute Pythagoras: $\\|S\\|_U^2 = \sum_r \left\\|f^{-1}(r)\cap S\right\\|_U^2$	$\\|\psi\\|^2 = \sum_\lambda \\|P_\lambda(\psi)\\|^2$
Normalized: $\sum_r \frac{\\|f^{-1}(r)\cap S\\|_U^2}{\\|S\\|_U^2} = \sum_r \frac{\|f^{-1}(r)\cap S\|}{\|S\|} = 1$	$\sum_\lambda \frac{\\|P_\lambda(\psi)\\|^2}{\\|\psi\\|^2} = 1$
Attribute Born rule: $\Pr(r\|S) = \frac{\\|f^{-1}(r)\cap S\\|_U^2}{\\|S\\|_U^2} = \frac{\|f^{-1}(r)\cap S\|}{\|S\|}$	$\Pr(\lambda\|\psi) = \frac{\\|P_\lambda(\psi)\\|^2}{\\|\psi\\|^2}$

Probability calculus for QM/sets over $\mathbb{Z}_2$ and for standard QM over $\mathbb{C}$

8 Measurement in QM/sets

8.1 Measurement, partitions, and distinctions

In QM/sets, numerical attributes $f : U \to \mathbb{R}$ can be considered as random variables on a set of equiprobable states $\{u\} \subseteq U$. The inverse images of attributes (or random variables) define set partitions $\{f^{-1}\} = \{f^{-1}(r)\}_{r\in f(U)}$ on the set U. Considered abstractly, the partitions on a set U are partially ordered by refinement where a partition $\pi = \{B\}$ *refines* a partition $\sigma = \{C\}$, written $\sigma \preceq \pi$, if for any block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$. The principal logical operation needed here is the *partition join* where the join $\pi \vee \sigma$ is the partition whose blocks are the non-empty intersections $B \cap C$ for $B \in \pi$ and $C \in \sigma$.

Each partition π can be represented as a binary relation $\operatorname{dit}(\pi) \subseteq U \times U$ on U where the ordered pairs (u, u') in $\operatorname{dit}(\pi)$ are the *distinctions* or *dits* of π in the sense that u and u' are in distinct blocks of π. These *dit sets* $\operatorname{dit}(\pi)$ as binary relations might be called *partition relations* (which are also called "apartness relations" in computer science). An ordered pair (u, u') is an *indistinction* or *indit* of π if u and u' are in the same block of π. The set of indits, $\operatorname{indit}(\pi)$, as a binary relation is just the equivalence relation associated with the partition π, the complement of the dit set $\operatorname{dit}(\pi)$ in $U \times U$.

In the category-theoretic duality between *sub*-sets (which are the subject matter of Boole's subset logic) and *quotient*-sets or partitions ([12] or [14]), the *elements* of a subset and the *distinctions* of a partition are corresponding concepts.[28]

[28] Boole has been included along with Laplace in the name of classical finite probability theory since he developed it as the normalized counting measure on the elements of the subsets of his logic. Applying the same mathematical move to the dual logic of partitions results in developing the notion of *logical entropy* $h(\pi)$ of a partition π as the normalized counting measure on the dit set $\operatorname{dit}(\pi)$, i.e., $h(\pi) = \frac{|\operatorname{dit}(\pi)|}{|U\times U|}$. ([11], [13])

The partial ordering of subsets in the Boolean lattice $\wp(U)$ is the inclusion of elements, and the refinement partial ordering of partitions in the partition lattice $\prod(U)$ is just the inclusion of distinctions, i.e., $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$. The top of the Boolean lattice is the subset U of all possible elements and the top of the partition lattice is the *discrete partition* $\mathbf{1} = \{\{u\}\}_{u \in U}$ of singletons which makes all possible distinctions: $\operatorname{dit}(\mathbf{1}) = U \times U - \Delta$ (where $\Delta = \{(u,u) : u \in U\}$ is the diagonal). The bottom of the Boolean lattice is the empty set $\emptyset$ of no elements and the bottom of the lattice of partitions is the *indiscrete partition* (or *blob*) $\mathbf{0} = \{U\}$ which makes no distinctions.

In the correspondences between QM/sets and QM, a block S in a partition on U [i.e., a vector $S \in \wp(U)$] corresponds to *pure* state in QM, and a partition $\pi = \{B\}$ on U is the *mixed state* of orthogonal pure states B. In QM, a measurement makes distinctions, i.e., makes alternatives distinguishable, and that turns a pure state into a mixture of probabilistic outcomes. The distinction-creating process of measurement in QM/sets is the action on S of the inverse-image partition $\left\{f^{-1}(r)\right\}_{r \in f(U)}$ in the join $\{S, S^c\} \vee \left\{f^{-1}(r)\right\}$ with the partition $\{S, S^c\}$, so that action on a given pure state S is:

$$S \longmapsto \left\{f^{-1}(r) \cap S\right\}_{r \in f(U)}$$

Action on the pure state S of an f-measurement-join
to give the mixed state $\left\{f^{-1}(r) \cap S\right\}_{r \in f(U)}$ on S.

The states $\left\{f^{-1}(r) \cap S\right\}_{r \in f(U)}$ are all possible or "potential" but the actual indefinite state S turns into one of the definite states with the probabilities given by the probability calculus: $\Pr(r|S) = \frac{\left\|f^{-1}(r) \cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(r) \cap S\right|}{|S|}$. Since the reduction of the state S to the state $f^{-1}(r) \cap S$ is mathematically described by applying the projection operator $f^{-1}(r) \cap ()$, it is called a *projective* measurement.

8.2 Imagery of measurement

It might be recalled that Hermann Weyl called a partition a "grating" or "sieve"[29] and then considered *both* set partitions and vector space partitions (direct sum decompositions) as the respective types of gratings.[47, pp. 255-257] He started with a numerical attribute on a set, e.g., $f : U \to \mathbb{R}$, which defined the set partition or "grating" [47, p. 255] with blocks having the same attribute-value, e.g., $\left\{f^{-1}(r)\right\}_{r \in f(U)}$. Then he moved to the QM case where the universe set, e.g., $U = \{u_1, ..., u_n\}$, or "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [47, p. 256]. The appropriate notion of a vector space partition or "grating" is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\overrightarrow{x}$ splits into r component vectors lying in the several subspaces" [47, p. 256], i.e., a direct sum decomposition of the space. After referring to a partition as a "grating" or "sieve," Weyl noted that "Measurement means application of a sieve or grating" [47, p. 259], e.g., in QM/sets, the application (i.e., join) of the set-grating or partition $\left\{f^{-1}(r)\right\}_{r \in f(U)}$ to the pure state $\{S\}$ to give the mixed state $\left\{f^{-1}(r) \cap S\right\}_{r \in f(U)}$.

For some mental imagery of measurement, we might think of the grating or sieve as a series of regular-polygonal-shaped holes that might shape an indefinite blob of dough. In a measurement, the blob of dough falls through one of the polygonal holes with equal probability and then takes on that shape.

[29] Arthur Eddington made a very early use of the sieve idea:

> In Einstein's theory of relativity the observer is a man who sets out in quest of truth armed with a measuring-rod. In quantum theory he sets out armed with a sieve.[10, p. 267]

This passage was quoted by Weyl [47, p. 255] in his treatment of gratings.

Figure 9: Measurement as randomly giving an indefinite blob of dough a definite polygonal shape.

A better image is a tree with a single root and many branches (think of the tree as a path from the indiscrete partition at the bottom of the lattice of partitions on U to the discrete partition top of the lattice). One might imagine each final branch as an specific eigen-address and a package traveling from the tree-root to a branch-address. But an address like S (e.g., just a zipcode) might be incomplete or indefinite between (i.e., a superposition of) all the eigen-addresses that lie beyond S.

Figure 10: Tree of branching outcomes

This allows us to address the question of what distinguishes the non-measurement evolution of a system (e.g., von Neumann's type 2 process of unitary evolution in full QM [46, p. 351]) from a measurement (von Neumann's type 1 process). If we think of the package with only the address S as starting at the root of the tree and traveling towards the branches, then at the first three-way branching, there is no need to distinguish between all the branches superposed to make S. The evolution or travel of the package is the same for all the addresses superposed in S; it doesn't make a difference. The route of the package is thus deterministically given (type 2 process). But when the package arrives at the branch-point that differentiates S, then a distinction has to be made to go any further because no matter which of the three branches is taken, it will exclude some of the final addresses–so that the branching "makes a difference" between the superposed alternatives in S.

Switching the imagery, consider a traveler who wanted to get to zipcode or region S and was otherwise indefinite as to the final address within the region. Then the traveler at each fork in the roadmap-tree outside S would deterministically take the fork in the tree that leads to S; there is no choice involved. That is the type 2 deterministic evolution. But once inside the region S, the traveler would have to flip a coin or roll a die at each fork in the road as that *makes a difference* as to the final address–which is like von Neumann's type 1 "measurement" process.

The imagery can be restated in terms of QM/sets. The tree is generated by a path from bottom to top is the lattice of partitions on U. Each partition along the path gives a refinement over the last partition on the path which is simply another branching of the tree. When measuring a pure state S using an attribute f, the result is the mixed state $\left\{f^{-1}(r) \cap S\right\}_{r \in f(U)}$ where S is assumed to be "chopped up" or differentiated into several distinct projected parts. If the returned eigenvalue is r, then there is a "type 1" state reduction or quantum leap of S to $f^{-1}(r) \cap S \neq S$ (i.e., von Neumann's type 1 process). But if the interaction represented by f "doesn't make a difference" in the sense that there is a specific r_0 such that $S \subseteq f^{-1}(r_0)$,[30] then S is not differentiated, $\Pr(r_0|S) = \frac{|f^{-1}(r_0) \cap S|}{|S|} = \frac{|S|}{|S|} = 1$, and there is no "type 1" quantum leap since $S = f^{-1}(r_0) \cap S$ (i.e., von Neumann's type 2 process). In this sense, the unitary evolution of a quantum system is the special case of a null measurement or in QM/sets, a no-distinction join where $f^{-1}(r_0) \cap S = S$.

In summary, there are two cases when a state S interacts with an apparatus or environment represented by f:

1. *vN's type 1 process*: the inverse-image partition $\left\{f^{-1}(r)\right\}_{r \in f(U)}$ chops up S into distinct parts $\left\{f^{-1}(r) \cap S\right\}_{r \in f(U)}$ so if the eigenvalue r is returned, then the state takes a "quantum leap" from S to $f^{-1}(r) \cap S$;

2. *vN's type 2 process*: there is, in effect, an eigenvalue r_0 such that $S \subseteq f^{-1}(r_0)$ so the "measurement" (i.e., interaction with the environment) makes "no difference" to S and thus there is no quantum leap since $S = f^{-1}(r_0) \cap S$, so the system follows a "no-distinction" evolution path that keeps the degree of distinction between states constant.[31]

Unitary evolution (or, equivalently, the Schrödinger equation) is often misinterpreted as a "law" analogous to the classical interpretation of Newton's laws. But it is only a description of the evolution of an isolated system with no distinction-making interactions.[32] The degree of distinctness or indistinctness of two quantum states ψ and φ in full QM is measured by their overlap $\langle\psi|\varphi\rangle$ so a "no-change-in-distinctness" evolution is one that preserves all overlaps, i.e., preserves the inner product, so a no-distinction evolution is a unitary evolution.

This characterization of a distinction-making "measurement" as opposed to a "no-distinction" evolution was made with some clarity long ago by Feynman.

> If you could, in principle, distinguish the alternative final states (even though you do not bother to do so), the total, final probability is obtained by calculating the probability for each state (not the amplitude) and then adding them together. If you cannot distinguish the final states even in principle, then the probability amplitudes must be summed before taking the absolute square to find the actual probability.[17, p. 3.9]

In full QM, when an evolving quantum system in an indefinite superposition state is coupled

[30] The first branching of the tree broke U into three blocks (the three branches) which could be considered as the inverse-image partition $\left\{f^{-1}\right\}$ of some attribute $f : U \to \mathbb{R}$. But since S was contained entirely in the middle branch, this is the condition $S \subseteq f^{-1}(r_0)$ for some r_0 where S is not "chopped up" or differentiated into several parts.

[31] There is one interesting case where type 1 and 2 processes overlap, namely, the second measurement "immediately" after a projective measurement that took S to $S \cap f^{-1}(r)$. For the *immediate* second "measurement," the initial state is $S \cap f^{-1}(r)$ which is contained in one of the eigenspaces $\wp\left(f^{-1}(r)\right)$ so the interaction involves making no distinctions in that initial state and thus it qualifies as a type 2 process as well (but which has no time to evolve to a different state).

[32] Interpreting the Schrödinger equation as a "law" that has to be obeyed is analogous to interpreting the mathematical description of the evolution of a thermodynamics system *in equilibrium* as being a "deterministic" law.

with another quantum-level system (e.g., neutrons scattering off atoms in a crystal)[33] or another macroscopic system, then the two types of processes depend on whether or not the normal interaction between the systems will "make a difference" as to the eigenstates in the superposition. If not, then normal no-distinction (= unitary) evolution ensues. If so, then a probabilistic "choice" must be made, and the resulting differential evolution constitutes a measurement with the outcome indicating the branch taken.

8.3 Example of a nondegenerate measurement

Our pedagogical methodology is to break controversial questions about full QM into two parts, the relatively simple question about QM/sets, and then the problem of translating the simplicities of QM/sets into full QM. One of the old questions is the mathematical form of the measurement process (since it cannot be described by a unitary transformation in QM or a nonsingular transformation in QM/sets).

In QM/sets, the distinction-making operation of "measurement" is mathematically described by the partition join operation. After seeing an example of a non-degenerate and degenerate QM/sets measurement using the partition join operation, we will reformulate the action of the partition join in QM/sets using the language of density matrices, and then we will see how that mathematical formulation of measurement as partition join in QM/sets is translated into full QM.

In the simple example illustrated below, we start at the one block or state of the indiscrete partition or blob which is the completely indistinct entity $\{a, b, c\}$. A measurement always uses some attribute that defines an inverse-image partition on $U = \{a, b, c\}$. In the case at hand, there are "essentially" four possible attributes that could be used to "measure" the indefinite entity $\{a, b, c\}$ (since there are four partitions that refine the indiscrete partition in Figure 3).

For an example of a nondegenerate measurement in QM/sets, consider any attribute $f : U \to \mathbb{R}$ which has the discrete partition as its inverse image (i.e., is injective), such as the ordinal number of the letter in the alphabet: $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. This attribute has three (nonzero) eigenvectors: $f \upharpoonright \{a\} = 1\{a\}$, $f \upharpoonright \{b\} = 2\{b\}$, and $f \upharpoonright \{c\} = 3\{c\}$ with the corresponding eigenvalues. The eigenvectors are $\{a\}$, $\{b\}$, and $\{c\}$, the blocks in the discrete partition of U. The nondegenerate measurement using the observable f acts on the pure state $U = \{a, b, c\}$ to give the mixed state of the discrete partition $\mathbf{1}$:

$$U \longmapsto \left\{U \cap f^{-1}(r)\right\}_{r=1,2,3} = \mathbf{1}.$$

Each such measurement would return an eigenvalue r with the probability of $\Pr(r|S) = \frac{|f^{-1}(r) \cap S|}{|S|} = \frac{1}{3}$ for $r \in f(U) = \{1, 2, 3\}$.

A projective measurement makes distinctions in the measured state that are sufficient to induce the "quantum jump" or projection to the eigenvector associated with the observed eigenvalue. If the observed eigenvalue was 3, then the state $\{a, b, c\}$ projects to $f^{-1}(3) \cap \{a, b, c\} = \{c\} \cap \{a, b, c\} = \{c\}$ as pictured below.

[33] Feynman gives a completely quantum level example of neutrons scattering off atoms in a crystal [17, Sec. 3-3] where in one case, it makes no difference (so no measurement) and in another case, it makes a difference (flipped spin in one of the atoms) so probabilities instead of amplitudes are added–even though such a "measurement" has nothing to do with a macroscopic apparatus, decoherence, and all that.

Figure 11: Nondegenerate measurement and resulting "quantum jump"

It might be emphasized that this is a state reduction from the single indefinite state $\{a,b,c\}$ to the single definite state $\{c\}$, not a subjective removal of ignorance as if the state had all along been $\{c\}$.

8.4 Example of a degenerate measurement

For an example of a degenerate measurement, we choose an attribute with a non-discrete inverse-image partition such as the partition $\pi = \{\{a\},\{b,c\}\}$. Hence the attribute could just be the characteristic function $\chi_{\{b,c\}}$ with the two eigenspaces $\wp(\{a\})$ and $\wp(\{b,c\})$ and the two eigenvalues 0 and 1 respectively. Since the eigenspace $\wp\left(\chi_{\{b,c\}}^{-1}(1)\right) = \wp(\{b,c\})$ is not one dimensional, the eigenvalue of 1 is a QM/sets-version of a *degenerate* eigenvalue. This attribute $\chi_{\{b,c\}}$ has four (non-zero) eigenvectors:

$$\chi_{\{b,c\}} \upharpoonright \{b,c\} = 1\{b,c\},\ \chi_{\{b,c\}} \upharpoonright \{b\} = 1\{b\},\ \chi_{\{b,c\}} \upharpoonright \{c\} = 1\{c\},\text{ and } \chi_{\{b,c\}} \upharpoonright \{a\} = 0\{a\}.$$

The "measuring apparatus" makes distinctions by joining the attribute inverse-image partition

$$\chi_{\{b,c\}}^{-1} = \left\{\chi_{\{b,c\}}^{-1}(1), \chi_{\{b,c\}}^{-1}(0)\right\} = \{\{b,c\},\{a\}\}$$

with the pure state representing the indefinite entity $U = \{a,b,c\}$. The action on the pure state is:

$$U \longmapsto \{U\} \vee \chi_{\{b,c\}}^{-1} = \chi_{\{b,c\}}^{-1} = \{\{b,c\},\{a\}\}.$$

The measurement of that attribute returns one of the eigenvalues with the probabilities:

$$\Pr(0|U) = \frac{|\{a\}\cap\{a,b,c\}|}{|\{a,b,c\}|} = \frac{1}{3} \text{ and } \Pr(1|U) = \frac{|\{b,c\}\cap\{a,b,c\}|}{|\{a,b,c\}|} = \frac{2}{3}.$$

Suppose it returns the eigenvalue 1. Then the indefinite entity $\{a,b,c\}$ reduces to the projected eigenstate $\chi_{\{b,c\}}^{-1}(1) \cap \{a,b,c\} = \{b,c\}$ for that eigenvalue [8, p. 221].

Since this is a degenerate result (i.e., the eigenspace $\wp\left(\chi_{\{b,c\}}^{-1}(1)\right) = \wp(\{b,c\})$ doesn't have dimension one), another measurement is needed to make more distinctions. Measurements by attributes, such as $\chi_{\{a,b\}}$ or $\chi_{\{a,c\}}$, that give either of the other two partitions, $\{\{a,b\},\{c\}\}$ or $\{\{b\},\{a,c\}\}$ as inverse images, would suffice to distinguish $\{b,c\}$ into $\{b\}$ or $\{c\}$. Then either attribute together with the attribute $\chi_{\{b,c\}}$ would form a *Complete Set of Compatible Attributes* or CSCA (i.e., the QM/sets-version of a Complete Set of Commuting Operators or CSCO [9]), where *complete* means that the join of the attributes' inverse-image partitions gives the discrete partition and where *compatible* means that all the attributes can be taken as defined on the same set of (simultaneous) basis eigenvectors, e.g., the U-basis.

Taking, for example, the other attribute as $\chi_{\{a,b\}}$, the join of the two attributes' partitions is discrete:

$$\chi_{\{b,c\}}^{-1} \vee \chi_{\{a,b\}}^{-1} = \{\{a\},\{b,c\}\} \vee \{\{a,b\},\{c\}\} = \{\{a\},\{b\},\{c\}\} = \mathbf{1}.$$

Hence all the eigenstate singletons can be characterized by the ordered pairs of the eigenvalues of these two attributes: $\{a\} = |0,1\rangle$, $\{b\} = |1,1\rangle$, and $\{c\} = |1,0\rangle$ (using Dirac's ket-notation to give the ordered pairs and listing the eigenvalues of $\chi_{\{b,c\}}$ first on the left).

The second projective measurement of the indefinite entity $\{b,c\}$ using the attribute $\chi_{\{a,b\}}$ with the inverse-image partition $\chi_{\{a,b\}}^{-1} = \{\{a,b\},\{c\}\}$ would have the pure-to-mixed state action:

$$\{b,c\} \longmapsto \left\{\{b,c\} \cap \chi_{\{a,b\}}(1), \{b,c\} \cap \chi_{\{a,b\}}(0)\right\} = \{\{b\},\{c\}\}.$$

The distinction-making measurement would cause the indefinite entity $\{b,c\}$ to turn into one of the definite entities of $\{b\}$ or $\{c\}$ with the probabilities:

$$\Pr(1|\{b,c\}) = \frac{|\{a,b\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2} \text{ and } \Pr(0|\{b,c\}) = \frac{|\{c\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2}.$$

If the measured eigenvalue is 0, then the state $\{b,c\}$ projects to $\chi_{\{a,b\}}^{-1}(0) \cap \{b,c\} = \{c\}$ as pictured below.

Figure 12: Degenerate measurement

The two projective measurements of $\{a,b,c\}$ using the complete set of compatible (e.g., both defined on U) attributes $\chi_{\{b,c\}}$ and $\chi_{\{a,b\}}$ produced the respective eigenvalues 1 and 0 so the resulting eigenstate was characterized by the eigenket $|1,0\rangle = \{c\}$.

Again, this is all analogous to standard Dirac-von-Neumann quantum mechanics.

8.5 Measurement using density matrices

The previous treatment of the role of partitions in measurement can be restated using density matrices [36, p. 98] over the reals. Given a partition $\pi = \{B\}$ on $U = \{u_1, ..., u_n\}$, the blocks $B \in \pi$ can be thought of as (nonoverlapping or "orthogonal") "pure states" where the "state" B occurs with the probability $p_B = \frac{|B|}{|U|}$. Then we can mimic the usual procedure for forming the density matrix $\rho(\pi)$ for the "orthogonal pure states" B with the probabilities p_B. The "pure state" B normalized in the reals to length 1 is represented by the column vector $|B\rangle_1 = \frac{1}{\sqrt{|B|}}[\chi_B(u_1), ..., \chi_B(u_n)]^t$(where $[]^t$ indicates the transpose). Then the *density matrix* $\rho(B)$ *for the pure state* $B \subseteq U$ is then (calculating in the reals):

$$\rho(B) = |B\rangle_1 (|B\rangle_1)^t = \frac{1}{|B|}\begin{bmatrix}\chi_B(u_1)\\ \chi_B(u_2)\\ \vdots \\ \chi_B(u_n)\end{bmatrix}[\chi_B(u_1), ..., \chi_B(u_n)]$$

$$= \frac{1}{|B|}\begin{bmatrix} \chi_B(u_1) & \chi_B(u_1)\chi_B(u_2) & \cdots & \chi_B(u_1)\chi_B(u_n) \\ \chi_B(u_2)\chi_B(u_1) & \chi_B(u_2) & \cdots & \chi_B(u_2)\chi_B(u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_B(u_n)\chi_B(u_1) & \chi_B(u_n)\chi_B(u_2) & \cdots & \chi_B(u_n) \end{bmatrix}.$$

For instance if $U = \{u_1, u_2, u_3\}$, then for the blocks in the partition $\pi = \{\{u_1, u_2\}, \{u_3\}\}$:

$$\rho(\{u_1, u_2\}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \rho(\{u_3\}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the "mixed state" *density matrix* $\rho(\pi)$ *of the partition* π is the weighted sum:

$$\rho(\pi) = \sum_{B\in\pi} p_B \rho(B).$$

In the example, this is:

$$\rho(\pi) = \frac{2}{3}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

While this construction mimics the usual construction of the density matrix for orthogonal pure states, the remarkable thing is that the entries have a direct interpretation in terms of the dits and indits of the partition π:

$$\rho_{jk}(\pi) = \left\{ \begin{array}{c} \frac{1}{|U|} \text{ if } (u_j, u_k) \in \text{indit}(\pi) \\ 0 \text{ if } (u_j, u_k) \notin \text{indit}(\pi) \end{array} \right. .$$

All the entries are real "amplitudes" whose squares are the two-draw probabilities of drawing a pair of elements from U (with replacement) that is an indistinction of π. As in the quantum case, the non-zero entries of the density matrix $\rho_{jk}(\pi) = \sqrt{\frac{1}{|U|}\frac{1}{|U|}} = \frac{1}{|U|}$ are the "coherences" [8, p. 302] which indicate that u_j and u_k "cohere" together in a block or "pure state" of the partition, i.e., $(u_j, u_k) \in \text{indit}(\pi)$. Since the ordered pairs (u_j, u_j) in the diagonal $\Delta \subseteq U \times U$ are always indits of any partition, the diagonal entries in $\rho(\pi)$ are always $\frac{1}{|U|}$.

Combinatorial theory gives a natural way to define the same density matrix of a partition. A binary relation $R \subseteq U \times U$ on $U = \{u_1, ..., u_n\}$ can be represented by an $n \times n$ *incidence matrix* $I(R)$ where

$$I(R)_{ij} = \left\{ \begin{array}{c} 1 \text{ if } (u_i, u_j) \in R \\ 0 \text{ if } (u_i, u_j) \notin R. \end{array} \right.$$

Taking R as the equivalence relation $\text{indit}(\pi)$ associated with a partition π, the density matrix $\rho(\pi)$ defined above is just the incidence matrix $I(\text{indit}(\pi))$ rescaled to be of trace 1 (i.e., sum of diagonal entries is 1):

$$\rho(\pi) = \frac{1}{|U|} I(\text{indit}(\pi)).$$

If the subsets $T \in \wp(U)$ are represented by the n-ary column vectors $[\chi_T(u_1), ..., \chi_T(u_n)]^t$, then the action of the projection operator $B \cap () : \wp(U) \to \wp(U)$ is represented by the $n \times n$ diagonal matrix P_B where the diagonal entries are:

$$(P_B)_{ii} = \left\{ \begin{array}{c} 1 \text{ if } u_i \in B \\ 0 \text{ if } u_i \notin B \end{array} \right. = \chi_B(u_i)$$

which is idempotent, $P_B^2 = P_B$, and symmetric, $P_B^t = P_B$. For any state $S \in \wp(U)$, the trace (sum of diagonal entries) of $P_B \rho(S)$ is:

$$\operatorname{tr}[P_B \rho(S)] = \frac{1}{|S|} \sum_{i=1}^{n} \chi_S(u_i) \chi_B(u_i) = \frac{|B \cap S|}{|S|} = \Pr(B|S)$$

so given $f : U \to \mathbb{R}$,

$$\Pr(r|S) = \frac{|f^{-1}(r) \cap S|}{|S|} = \operatorname{tr}\left[P_{f^{-1}(r)} \rho(S)\right].$$

We saw previously how the action of a measurement in QM/sets could be described using the partition join operation. The join $\pi \vee \sigma$ of the partitions $\pi = \{B\}$ and $\sigma = \{C\}$ could be seen as the result $\bigcup_{C \in \sigma} \{C \cap B \neq \emptyset : B \in \pi\}$ of the projection operators $C \cap ()$ acting on the $B \in \pi$ for all $C \in \sigma$. Substituting the normalized $|B\rangle_1$ for B with the density matrix $\rho(B) = |B\rangle_1 (|B\rangle_1)^t$ and the matrix projection operators P_C for $C \cap ()$, the application of P_C to $|B\rangle_1$ yields the density matrix:

$$(P_C |B\rangle_1)(P_C |B\rangle_1)^t = P_C |B\rangle_1 (|B\rangle_1)^t P_C^t = P_C \rho(B) P_C.$$

Summing with the probability weights gives: $\sum_{B \in \pi} p_B P_C \rho(B) P_C = P_C \rho(\pi) P_C$ and then summing over the different projection operators gives: $\sum_{C \in \sigma} P_C \rho(\pi) P_C$. A little calculation then shows that this is exactly the *density matrix of the partition join*:

$$\sum_{C \in \sigma} P_C \rho(\pi) P_C = \rho(\pi \vee \sigma).$$

Density matrix version of the partition join

We are modeling, using density matrices, the QM/sets projective measurement of an attribute $f : U \to \mathbb{R}$ starting with a pure state S. The measurement converts the pure state $|S\rangle$ to one of the states $|f^{-1}(r) \cap S\rangle$ with the probability $\frac{|f^{-1}(r) \cap S|}{|S|}$. In the previous example of a (degenerate) measurement with $U = \{a, b, c\}$ and $f = \chi_{\{b,c\}}$, then the measurement, in terms of partitions, had the effect of making distinctions on the partition $\{U\}$ by the partition $\{f^{-1}\} = \left\{\chi_{\{b,c\}}^{-1}\right\}$using the join operation:

$$\{U\} \longmapsto \{U\} \vee \left\{\chi_{\{b,c\}}^{-1}\right\} = \{\{b, c\}, \{a\}\}.$$

The mixed state $\{\{b, c\}, \{a\}\}$ has the projected outcomes $\chi_{\{b,c\}}^{-1}(1) \cap U = \{b, c\}$ and $\chi_{\{b,c\}}^{-1}(0) \cap U = \{a\}$ which occur with the probabilities $\Pr(1|U) = \left|\chi_{\{b,c\}}^{-1}(1) \cap U\right| / |U| = 2/3$ and $\Pr(0|U) = \left|\chi_{\{b,c\}}^{-1}(0) \cap U\right| / |U| = 1/3$.

We now have the density matrix version of the partition join operation, so in the general case of starting with the pure state S, we might take the starting partition on U as $\pi = \{S, S^c\}$ and then take the measurement join with $\sigma = \{f^{-1}\} = \{f^{-1}(r)\}_{r \in f(U)}$ which yields the density matrix (using linearity):

$$\rho\left(\{S, S^c\} \vee \{f^{-1}\}\right) = \sum_{r \in f(U)} P_{f^{-1}(r)} \rho(\{S, S^c\}) P_{f^{-1}(r)}$$
$$= p_S \sum_{r \in f(U)} P_{f^{-1}(r)} \rho(S) P_{f^{-1}(r)} + p_{S^c} \sum_{r \in f(U)} P_{f^{-1}(r)} \rho(S^c) P_{f^{-1}(r)}.$$

Thus starting with the pure state density matrix $\rho(S) = |S\rangle_1 (|S\rangle_1)^t$, the action of the measurement given by the partition join (ignoring the action on the complement S^c) is to create the mixed state $\hat{\rho}(S)$:

$$\boxed{\rho(S) \longmapsto \hat{\rho}(S) = \sum_{r \in f(U)} P_{f^{-1}(r)} \rho(S) P_{f^{-1}(r)}}$$

Action of measurement of attribute f on the pure state density matrix $\rho(S)$.

In that mixed state, the projected state $\left|f^{-1}(r) \cap S\right\rangle$ occurs with the probability $\operatorname{tr}[P_{f^{-1}(r)}\rho(S)] = \frac{|f^{-1}(r)\cap S|}{|S|} = \Pr(r|S)$.

In full QM, the projective measurement using a Hermitian observable operator L with the spectral decomposition $L = \sum_{i=1}^{m} \lambda_i P_i$ of a normalized pure state $|\psi\rangle$ results in the state $P_i|\psi\rangle$ with the probability $p_i = \operatorname{tr}[P_i\rho(\psi)] = \Pr(\lambda_i|\psi)$ where $\rho(\psi) = |\psi\rangle\langle\psi|$. The projected resultant state $P_i|\psi\rangle$ has the density matrix $\frac{P_i|\psi\rangle\langle\psi|P_i}{\operatorname{tr}[P_i\rho(\psi)]} = \frac{P_i\rho(\psi)P_i}{\operatorname{tr}[P_i\rho(\psi)]}$ so the mixed state describing the probabilistic results of the measurement is [36, p. 101 or p. 515]:

$$\hat{\rho}(\psi) = \sum_i p_i \frac{P_i\rho(\psi)P_i}{\operatorname{tr}[P_i\rho(\psi)]} = \sum_i \operatorname{tr}[P_i\rho(\psi)] \frac{P_i\rho(\psi)P_i}{\operatorname{tr}[P_i\rho(\psi)]} = \sum_i P_i\rho(\psi)P_i.$$

Thus we see how the density matrix treatment of measurement in QM/sets is just a sets-version of the density matrix treatment of projective measurement in standard Dirac-von-Neumann QM. And we have the additional interpretation-relevant information that the measurement is described by the distinction-creating partition join operation in QM/sets. That confirms the observation in QM that the essence of measurement is distinguishing the alternative possible states–which in this interpretation means taking an objectively indefinite state to a more definite state. And the "detour" through the QM/sets description of measurement using partition joins provides the interpretive "back-story" for the already-known mathematical description of projective measurement in full QM:

$$\rho(\psi) \longmapsto \hat{\rho}(\psi) = \sum_i P_i\rho(\psi)P_i.$$

9 Conclusions

To develop the objective indefiniteness interpretation of QM, our strategy was to "follow the math." Mathematics tells us that there are two dual visions of reality: the definite-all-the-way-down vision assumed in common sense and used in classical physics, and the notion of an indefinite reality used in quantum physics, i.e., in category theory, subsets versus quotient sets, or limits versus colimits. The two notions were illustrated by the two ways to interpret a set $\{a, c\}$ as either a set of two fully definite entities a and c, or as a single indefinite "superposition" entity that is indefinite between the definite entities a and c.

Following the math to understand indefiniteness means studying the mathematical way to describe indefiniteness which is to start with some universe of definite eigen-alternatives, and then to quotient out some "surplus" definiteness so the blocks (or equivalence classes) in the resulting partition, like $\{a, c\}$ in the partition $\{\{b\}, \{a, c\}\}$ of the universe $U = \{a, b, c\}$, would represent indefiniteness between the definite alternatives in the block. This partitional analysis was developed both at the simple level of sets and at the level of the complex vector spaces used in QM. Partitions were shown to be defined for both sets and complex vector spaces in a top-down fashion by attributes/observables, and in a bottom-up fashion by the representations of symmetry groups.

The quantum probability calculus was developed for sets in the pedagogical model of QM/sets where it is shown to be a non-commutative verion of the usual Laplace-Boole finite probability theory. The parallels were emphasized with the usual quantum probability calculus in the complex vector spaces of ordinary Dirac-von-Neumann QM. Finally, the controversial topic of measurement was analyzed in QM/sets and shown to be entirely analogous to the usual treatment in full QM–with some additional information about making distinctions using the joins of set or vector space partitions.

In both QM/sets and full QM, the mathematical machinery is thus shown to revolve around partitions, and partitions are the mathematical way to describe indefiniteness. In this manner, following the math suggests that the realistic (i.e., not merely epistemic) way to interpret quantum mechanics is what Abner Shimony called the Literal or objective indefiniteness interpretation.

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