On Vectorial Marginal Products and Modern Property Theory

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Abstract

Neoclassical economic theory presents marginal productivity (MP) theory using the scalar notion of marginal products, and takes pains, implicitly or explicitly, to show that competitive equilibrium satisfies the supposedly ethical principle: "To each what he and the instruments he owns produces." This paper shows that MP theory can also be formulated in a mathematically equivalent way using vectorial marginal products–which however conflicts with the above-mentioned "distributive shares" picture. Vectorial MP theory also facilitates the presentation of modern property theory which on the descriptive side is based on the fact that, contrary to the distributive shares picture, one legal party owes 100 percent of the liabilities for the used-up inputs and owns 100 percent of the produced outputs in a productive opportunity. On the normative side, modern property theory is the old "labor theory of property" presented in the modern form as the juridical-ethical principle of imputing legal responsibility in accordance with de facto responsibility for the liabilities and assets created in production–where, of course, only persons and not things ("the instruments he owns") have responsible agency. Vectorial marginal products (with positive and negative components) thus facilitates presenting the actual ethical principle: "To each *person* the assets *and liabilities* he or she produces (usually jointly with other persons)."

Keywords: marginal productivity theory; property theory; imputation of responsibility

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1 Introduction

When an orthodox economist considers the principle of people getting the fruits of their labor, he or she will invariably interpret it in terms of marginal productivity. The orthodox claim is that under the conditions of competitive equilibrium, each unit of labor "gets what it produces." Indeed, Milton Friedman calls it the "capitalist ethic" [6, p. 164]:

> The ethical principle that would directly justify the distribution of income in a free market society is, "To each what he and the instruments he owns produces." [6, pp. 161-2]

Well-meaning liberal, progressive, and even heterodox economists emphasize that the actual economy may be neither competitive nor in equilibrium, and in any case, there are enormous difficulties in measuring the marginal product of each factor of production. They raise no objection in principle to the interpretation of marginal productivity theory as giving people "what they produce" but they fuss about its applicability in practice as well as about the prior personal distribution of factor ownership.[1]

It has been argued ([2], [3]) that employer-employee system does not satisfy, even in principle, the norm of people getting the fruits of their labor. The orthodox view of marginal productivity theory is flawed on several counts. Firstly, non-human "agents of production" do not have responsible agency so they cannot be responsible for anything. Tools and machines do not "produce" their marginal product or anything else. Tools and machines are used by people to produce the outputs, and the using-up of the services of the capital is the negative fruits of their labor. Secondly, the shares in the product are not actually imputed or assigned to the various factor suppliers. In terms of property rights, one legal party appropriates the "whole product" of a firm, 100 percent of the output assets and 100 percent of the input liabilities. Modern economists display a learned ignorance of this simple legal fact since they have learned the "distributive shares" framing of the question. It seems one has to go back to before the development of MP theory to find a statement of the simple legal fact that one party covers all the liabilities for "both instruments of production" capital and labor, and owns the whole of the outputs.

> In the state of society, in which we at present exist, it is in these circumstances that almost all production is effected: the capitalist is the owner of both instruments of production: and the whole of the produce is his. [10, Chapter I, section II]

There is still another flaw in the orthodox treatment of MP theory and that is our principal topic. The ideological baggage being carried by MP theory forces it to be presented in a factually implausible way. The factually implausible part of the orthodox view is the picture of a unit of a factor as producing its marginal product *ex nihilo* (even assuming we personify the factors with responsible agency). The idea is that when another unit of an input x is used, then by considering a hypothetical shift to a slightly more x-intensive production process, more output can be produced using the same amount of the other factors, and that extra output is the "marginal product" of that x factor. But cost-minimizing firms would not make such a hypothetical shift when increasing factor usage; they would expand along the least-cost expansion path. More of other factors must then be used, so the additional factors and product would be given by a vector, a *vectorial marginal product*.

[1] See, for instance, Chapter 6, "To each according to his contribution" in Keen [8].

When the value of these used-up factors is subtracted from the value of the extra product, the result is the marginal profit of an extra unit of x, not the "value of the marginal product" of the unit of the factor x. More to the point, the simple "distributive shares of the product" framing collapses since the vectorial marginal product includes both a "share" of the output but also an increase in using up other inputs to expand along the least-cost expansion path.

In this essay, we give the mathematically equivalent vectorial presentation of MP theory, which is based on the more plausible picture that a unit of labor can only produce more of the outputs by using up more of the other inputs at minimum cost. The "problem" with this version of MP theory is that it does not lend itself to the ideologically appealing picture of each unit of a factor as "producing its marginal product." Thus we have a central example about how the ideological baggage being towed by orthodox economics affects even the mathematical presentation of the standard theories.

2 The Conventional Picture of Scalar Marginal Products

Marginal productivity (MP) theory has always played a larger importance in orthodox economics than could be justified by its purely analytical role. This is because MP theory is conventionally interpreted as showing that, in competitive equilibrium, "each factor gets what it is responsible for producing." The marginal unit of a factor is seen as producing the marginal product of that factor, and each unit could be taken as the marginal unit, so each unit "produces its marginal product." Consider the *marginal physical product of labor* MP_L. In competitive equilibrium, the value of the marginal product of labor pMP_L (where p is the unit price of the output) is equal to w (the unit price of labor):

$$pMP_L = w$$

"Value of what a unit produces" = "Value received by a unit of the factor."

There are many problems in this conventional interpretation of MP theory. Our purpose is to highlight an internal incoherence in the conventional treatment, to show how this difficulty can be overcome in a mathematically equivalent reformulation of MP theory, and to note how this reformulation accommodates a rather different interpretation of the theory.

The problem (or internal incoherence) in the usual treatment is simply that a unit of a factor cannot produce its marginal product out of nothing. The factor must simultaneously use some of the other factors. If the marginal product of one man-year in a tractor factory is one tractor, how can a tractor be produced without using steel, rubber, energy, and so forth? But when that concurrent factor usage is taken into account ("priced out"), then the usual equations must be significantly reformulated. A new vectorial notion of the marginal product, the "vector marginal product," must be used in place of the conventional scalar marginal product.

Before turning to the vectorial treatment of marginal products we must remove the seeming paradox in the scalar treatment. When we increase the labor in a tractor factory to produce more tractors, we will also have to increase the steel, rubber, energy, and other inputs necessary to produce tractors. That would spoil the attempt to take the increase in tractor output as the result of solely the increase in labor. But the so-called "marginal product of labor" is the result of a somewhat different hypothetical or conjectural change in production. It is assumed that factors are substitutable. To arrive at the "marginal product of labor" we must consider two changes: an increase in labor and a shift to a slightly more labor-intensive production technique so that the increased labor can be used together with exactly the same total amounts of the other factors. Since (following the hypothetical production shift) the other factors are used in the same total amounts, the extra output is then viewed as the "product" of the extra unit of labor, as if the extra product was produced *ex nihilo* by the extra unit of labor.

There is one other point that might be mentioned. Since the usual "story" represents each factor as producing a share of the product (incurring no other costs) and getting the value of that share, one might wonder if there is a "dual story" about the distribution of the costs. Indeed, there is. The metaphorical picture of each input-supplier as "producing" a share of the outputs through "the instruments" supplied, dualizes to the picture of each output-demander as using up a share of the inputs consumed in producing the unit of output demanded. The value of those used-up inputs at the margin is the marginal cost MC so the dual part of the "capitalist ethic" is that the output-demander should owe for those liabilities and thus pay the price $p = MC$. But in terms of non-metaphorical property assets and liabilities, the input-suppliers do not own shares of the output assets, and the output-demanders do not owe for a share of the total input liabilities. There is one legal party (sometimes called the "residual claimant" or simply the "firm") which stands between the input suppliers and output demanders, and as already noted, that legal party appropriates 100 percent of the input liabilities and 100 percent of the output assets, i.e., legally appropriates what will here be called the "whole product." The fundamental question about production is not about distributive shares in asset or liability values, but the prior question: "Who is to be the firm: Capital, Labor, or the State?" That is the question addressed by property theory, not value theory.

3 Symmetry Restored: The Pluses and Minuses of Production

Nothing is produced *ex nihilo*. Labor cannot produce tractors without actually using other inputs. Production needs to be reconceptualized in an algebraically symmetric manner. That is, there are both positive results (produced outputs) and negative results (used-up inputs), and they can be considered symmetrically in a vectorial form.

For a nontechnical presentation, let $Q = f(K, L)$ be a production function with p, r, and w as the unit prices of the outputs Q, the capital services K, and the labor services L respectively. The outputs Q are the positive product of production but there is also a negative product, namely the used-up capital and labor services K and L. Lists or vectors with three components can be used with the outputs, capital services, and labor services listed in that order. The *positive product* would be represented as $(Q, 0, 0)$. The *negative product* signifying the used-up or consumed inputs could be represented as $(0, -K, -L)$. The comprehensive and algebraically symmetric notion of the product is obtained as the (component-wise) sum of the positive and negative products. It might be called the *whole product* [where symbols for vectors are in bold].

$$\mathbf{WP} = (Q, -K, -L) = (Q, 0, 0) + (0, -K, -L)$$
$$\text{Whole Product} = \text{Positive Product} + \text{Negative Product}$$

The unit prices can also be arranged in a vector, the *price vector* $\mathbf{P} = (p, r, w)$. The (dot) product of a price vector times a quantity vector (such as the whole product vector) is the sum of the component-wise products of prices times quantities. That sum is the value of the quantity vector.

$$\mathbf{P}\cdot(Q, 0, 0) = (p, r, w) \cdot (Q, 0, 0) = pQ$$
$$\text{Value of Positive Product} = \text{Revenue}$$
$$\mathbf{P} \cdot (0, -K, -L) = (p, r, w) \cdot (0, -K, -L) = -(rK + wL)$$
$$\text{Value of Negative Product} = \text{Expenses}$$
$$\mathbf{P} \cdot (Q, -K, -L) = (p, r, w) \cdot (Q, -K, -L) = pQ - (rK + wL)$$
$$\text{Value of Whole Product} = \text{Profit}$$

4 Marginal Whole Products

The alternative presentation of MP theory uses the marginal version of the whole product, which we will call the "marginal whole product."[2] The precise mathematical development is given later. Here we develop a heuristic discrete treatment. Given the input prices and a given level of output Q_0, there are input levels K_0 and L_0 that produce Q_0 at minimum cost $C_0 = rK_0 + wL_0$.

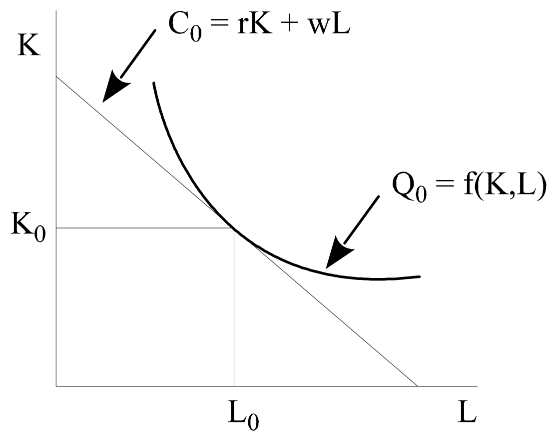

Figure 1: Minimum Cost to Produce Quantity Q_0

For an increase of one unit of output to $Q_1 = Q_0 + 1$, there will be new levels of K_1 and L_1 necessary to produce Q_1 at minimum cost.

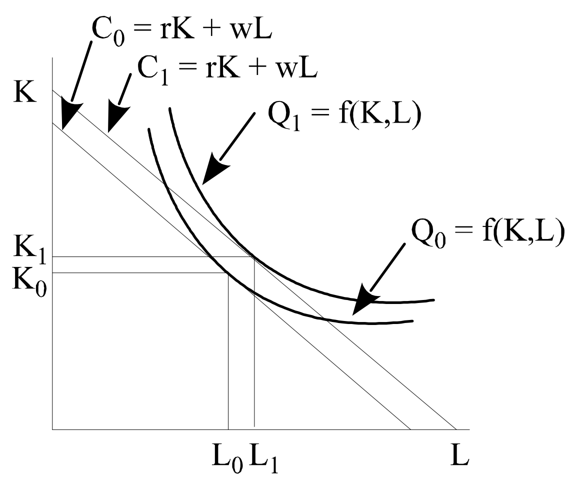

Figure 2: New Levels of K and L to Produce $Q_1 = Q_0 + 1$

Let $\Delta K = K_1 - K_0$ and $\Delta L = L_1 - L_0$ be the marginal increases in the amounts of capital and labor services that are necessary to produce the increase in output $\Delta Q = Q_1 - Q_0 = 1$. The minimum cost of producing Q_1 is $C_1 = rK_1 + wL_1$. Since $\Delta Q = 1$, the marginal cost is:

$$MC = \Delta C/\Delta Q = \Delta C = C_1 - C_0 = rK_1 + wL_1 - (rK_0 + wL_0).$$

[2] Generally, the adjective "whole" will indicate a vectorial notion.

The marginal version of the whole product is the *marginal whole product* which has unit output and minus the inputs necessary to produce one more unit of output at minimum cost.

$$\mathbf{MWP} = (1, -\Delta K, -\Delta L)$$
Marginal Whole Product

The value of the marginal whole product is the *marginal profit*, the difference between price and marginal cost.

$$\mathbf{P} \cdot \mathbf{MWP} = (p, r, w) \cdot (1, -\Delta K, -\Delta L) = p - [(rK_1 + wL_1) - (rK_0 + wL_0)] = p - MC.$$
Value of Marginal Whole Product is Marginal Profit

If the marginal profit was positive at a given level of output, then profits could be increased by increasing the level of output. If the marginal profit was negative, then profits would increase by decreasing the level of output. Thus if profits are at a maximum, then the marginal profit must be zero. This is the usual result that $p = MC$ if profits are at a maximum.

5 Asymmetry Between Responsible and Non-Responsible Factors

Part of the poetic charm of the conventional presentation of MP theory was that it allowed each factor to be pictured as active—as being "responsible" for producing its own marginal product. But we have noted the technical absurdity of, say, labor producing tractors out of nothing else. Labor must use up steel, rubber, and other inputs to produce tractors. But if that is accepted, then it is implausible to turn around and pretend that another factor is also active—that steel uses up labor, rubber, and other factors to produce tractors. Hence neoclassical economics uses the usual picture of each factor as "producing its marginal product" without using other factors–which then allows it to invoke "the ethical proposition that an individual deserves what is produced by the resources he owns" [5, p. 199].

MP theory, as an analytical economic theory, does not provide any distinction between responsible or non-responsible factors. Those notions must be imported. No amount of staring at partial derivatives will reveal the difference between responsible and non-responsible factors. "Responsibility" is a legal-jurisprudential notion. Neoclassical theory uses poetic license and the pathetic fallacy to represent all the factors as being responsible and cooperating together to produce the outputs. For instance, "Together, the man and shovel can dig my cellar" or "land and labor together produce the corn harvest" [11, pp. 536-7]. But poetry aside, a man uses a shovel to dig a cellar and people use land (and other inputs) to produce the corn harvest. Only human actions can be responsible for anything. For example, the tools of the burglary trade certainly have a causal efficacy ("productivity"), but only the burglar can be charged with responsibility for the crime. The responsibility is imputed back through the tools (as "responsibility conduits") to the human user.

The legally-trained Austrian economist, Friedrich von Wieser, introduced the notion of imputation into economics to metaphorically talk about the "responsible agency" of all the agents of production. But even he was quite clear that for the non-metaphorical notions of legal or moral imputation, only persons could be responsible.

> The judge ... who, in his narrowly-defined task, is only concerned with the legal imputation, confines himself to the discovery of the legally responsible factor,–that person, in fact, who is threatened with the legal punishment. On him will rightly be laid the whole burden of the consequences, although he could never by himself alone–without instruments and all the other conditions–have committed the crime. The imputation takes for granted physical causality. ...

> If it is the moral imputation that is in question, then certainly no one but the labourer could be named. Land and capital have no merit that they bring forth fruit; they are dead tools in the hand of man; and the man is responsible for the use he makes of them. [13, pp. 76-79]

This admission about non-metaphorical legal and moral imputation was only made in the early days of establishing the metaphorical reinterpretation of imputation used in the usual presentation of MP theory.

> In the division of the return from production, we have to deal similarly ... with an imputation, – save that it is from the economic, not the judicial point of view. [13, p. 76]

The modern texts just present the metaphorical "economic" imputation as if it was the only imputation.

Modern property theory deals with non-metaphorical legal and moral imputation precisely from "the judicial point of view." Property theory [3] on its normative side was traditionally called the "labor theory of property"[3] but in its modern form, it is the *juridical principle of imputation*: Assign legal rights and liabilities to the de facto responsible agents [2]. For instance, only the persons (including managers) working in a productive opportunity are de facto responsible for using up the inputs in the process of producing the outputs, and thus they should jointly have those legal liabilities (used-up inputs = negative fruits of their labor) and have the legal ownership of those assets (produced outputs = positive fruits of their labor), i.e., they should jointly appropriate the *whole product of labor* which can be seen as the whole product $\mathbf{WP} = (Q, -K, -L)$ plus the labor services $(0, 0, L)$ viewed as a commodity (that they create and use up).

$$\mathbf{WP}_L = (Q, -K, 0) = \mathbf{WP} + (0, 0, L)$$

Whole product of labor.

Since MP theory does not, by itself, provide any concept of "responsible" factors, any factor or factors could be taken as the responsible factors for analytical purposes. In the mathematical treatment given below, the factors $x_1, ..., x_n$ will not be identified (as capital, labor, etc.), and we will arbitrarily take the first factor as being responsible. In our nontechnical presentation where the factors are identified, labor will taken as the responsible factor (but the *formalism* would be the same, *mutatis mutandis*, for any other choice).

As the responsible factor produces the outputs (produces the positive product), it must also use up the inputs (produce the negative product). We must calculate the positive and negative product of the marginal unit of the responsible factor, labor. We will call the vector of positive and negative marginal results of labor, the "marginal whole product of labor." The marginal whole product of labor is then compared with the opportunity cost of labor (the wage w in the model).

The marginal quantities $\Delta Q = 1$, ΔK, and ΔL that appear in the marginal whole product can be used to form the ratios $\Delta Q/\Delta K$ and $\Delta Q/\Delta L$. But these ratios are *not* the marginal products. For instance, if labor is increased by this ΔL, then an additional ΔK must be used up to produce one more unit of output ($\Delta Q = 1$) in a cost-minimizing manner. The usual "marginal product" of labor MP_L is the extra product produced per extra unit of labor if the production technique is hypothetically shifted so that no more extra capital is used.

[3] See, for instance, Thomas Hodgskin [7] and Foxwell's introduction [4] to the book by Carl Menger's legally-trained brother, Anton Menger [9].

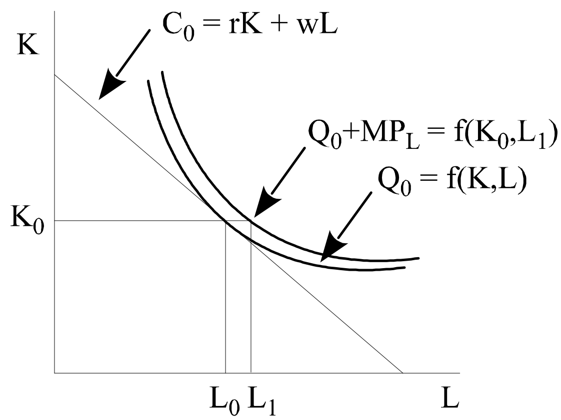

Figure 3: MP_L as the increase in Q from ΔL keeping capital constant at K_0.

In our simple model, the marginal results of labor can be calculated by dividing the marginal whole product through by ΔL to obtain $(1/\Delta L, -\Delta K/\Delta L, -1)$. Since labor also creates the marginal unit of labor $(0,0,1)$, the marginal whole product of labor is the following vector sum.

$$\mathbf{MWP_L} = (1/\Delta L, -\Delta K/\Delta L, 0) = (1/\Delta L, -\Delta K/\Delta L, -1) + (0,0,1).$$

Marginal Whole Product of Labor

Multiplying through by the prices yields the corresponding value.

$$\mathbf{P} \cdot \mathbf{MWP_L} = (p,r,w) \cdot (1/\Delta L, -\Delta K/\Delta L, 0) = (p - r\Delta K)/\Delta L = w + (p - MC)/\Delta L.$$

Value of Marginal Whole Product of Labor

If $\mathbf{P} \cdot \mathbf{MWP_L}$ (the net value of the fruits of the marginal unit of labor) exceeds w (the opportunity cost of the marginal unit of labor), then it is profitable to increase the amount of labor to produce more output by using up more capital services. Conversely, if $\mathbf{P} \cdot \mathbf{MWP_L}$ is less then w, then the use of the marginal unit of labor does not cover its opportunity cost so it would be better to reduce the amount of labor. Thus for profits to be maximized, the value of the marginal whole product of labor must equal the opportunity cost of labor.

$$\mathbf{P} \cdot \mathbf{MWP_L} = w.$$

Profit Max Implies: Value of Marginal Whole Product of Labor = Wage

Since $\mathbf{P} \cdot \mathbf{MWP_L} = w + (p - MC)/\Delta L$, the above result is equivalent to the usual $p = MC$.

6 Comparison of the Two Treatments of MP Theory

We have given an alternative treatment of MP theory using vector marginal products. We have also used the juridical notion of the responsible factor (here taken as labor) to organize the presentation. The crux of the two presentations is in the two marginal conditions concerning labor:

Conventional labor equation:	$p \cdot MP_L = w$
Alternative labor equation:	$\mathbf{P} \cdot \mathbf{MWP_L} = w.$

In the conventional labor equation, p and MP_L (as well as w) are scalars. In the alternative equation, $\mathbf{P}$ and $\mathbf{MWP_L}$ are vectors (while w remains a scalar). The conventional interpretation of MP_L pictures labor as producing marginal products without using up any inputs ("virgin birth of marginal products"). The marginal whole product of labor $\mathbf{MWP_L}$ gives the picture of the marginal effect of labor as producing outputs by using up other inputs at minimum costs.

Since the alternative presentation gives a more realistic treatment of marginal production, one might ask why it isn't used. One "problem" in the alternative treatment is that it does not allow the symmetrical picture of each factor as "producing" its share of the outputs. Since conventional production is based on all factors being treated symmetrically as being legally rentable, it is inconvenient to have a theory that suggests an alternative arrangement.

One could, of course, take capital services as the active or responsible factor, define the marginal whole product of capital as $\mathbf{MWP_K} = (1/\Delta K, 0, -\Delta L/\Delta K)$, and then show that the following condition is also equivalent to profit maximization (when costs are minimized).

$$\mathbf{P} \cdot \mathbf{MWP_K} = r$$

Profit Max Implies: Value of Marginal Whole Product of Capital = Rental

But instead of restoring a peaceful symmetry, this only highlights the conflict since one cannot plausibly represent both capital as producing the product by using labor, and labor as producing the product by using capital. MP theory itself provides no grounds for choosing one of the conflicting pictures over the other—for choosing the picture of the burglar using tools to commit the crime over the picture of the tools using the burglar to commit the crime. The distinction between the two pictures comes from jurisprudence, not from economics.

The conventional treatment of MP theory is clearly superior in terms of a "symmetrical" treatment of persons and things. The marginal unit of each factor can be presented as immaculately producing its marginal product. The same picture can be used for each factor without any conflict.

Since the alternative treatment that acknowledges that marginal products cannot be produced *ex nihilo* seems superior on analytical grounds (don't neoclassical firms expand along the least-cost expansion path?), orthodox economics would indeed seem to choose the conventional treatment of MP theory over the mathematically equivalent vectorial treatment as a "pre-analytical judgment."

7 Conclusion: The "Advantages" of Scalar MP Theory

There are two basic problems with the usual scalar MP theory "story." One problem is that no factor x_i can produce a part of the product without using up other factors, and the usual "story" only seems that way by assuming a conjectural shift to a more x_i-intensive production process so that $MP_i = \partial y/\partial x_i$ more units are produced using the extra x_i and the same total amounts of the other factors. But that is only a fictitious story since the neoclassical firm would only expand output along the least-cost expansion path which would involve using up increments in the other factors along with the increase in some input x_i. Hence the non-fictitious notion of the marginal product of an additional unit of the x_i is the *vectorial* marginal product of the factor that computes all the marginal changes along the least-cost expansion path. Why doesn't the economics profession use the mathematically equivalent theory with vector marginal products describing motion along the least-cost expansion path? But then the whole framing about distributive shares in the outputs collapses since it ignores the liabilities also created in production and it ignores the actual appropriation of 100% of the new property assets and liabilities (the whole product) going to one party in a productive opportunity.[4] Thus it seems the fictitious story about "distributive shares" based on scalar marginal product theory was not a "bug" but a "feature" of the theory.

That brings us to the second problem in the usual presentation of MP theory. As von Wieser put it:

[4]In spite of obscene distribution of income and wealth resulting from centuries of leveraging or renting human beings, even the most progressive of neoclassical economists (e.g., Stiglitz, Krugman, and Piketty) continue to frame the question in terms of the distributive shares picture–while ignoring the prior question of who is to "be the firm" (i.e., who is to appropriate the whole product) in the first place.

"no one but the labourer could be named. Land and capital have no merit that they bring forth fruit; they are dead tools in the hand of man; and the man is responsible for the use he makes of them." [13, p. 79]

Only the actions of the persons involved in a production opportunity can be de facto responsible for the results, and the results are algebraically symmetric, i.e., both negative and positive. In general terms, Labor L (human actions of all the people working in a firm) uses up the services of capital K (including land) to produce the outputs Q. And since those human actions are represented as an input L, Labor also produces L and uses up L in the productive process. Hence the "whole" (both positive and negative) product of the responsible factor L is the whole product of labor vector $\mathbf{WP_L} = (Q, -K, 0)$ which can be represented as the whole product $(Q, -K, -L)$ plus the labor services $(0, 0, L)$. Hence the real Adding Up Theorem integrates the marginal whole product of the responsible factor L to see what that factor is responsible for *in toto*–as we did for the Cobb-Douglas production function. And by the fundamental theorem of the calculus, the integration of $\mathbf{MWP_L}$ is equivalent to computing the net difference between when the people working in the productive process carry out the actions L and when they do nothing: $(Q, -K, 0) - (0, 0, 0) = \mathbf{WP_L}$.[5]

The bigger "problem" with using vectorial marginal products and using the standard juridical fact that only persons can be responsible for anything is that it does not give a satisfactory "account" of the standard employment system where the people working in the productive process are rented,[6] hired, or employed. Under the employment system, the employees are only recognized as owning their labor $(0, 0, L)$ which is sold in the employer-employee relationship to the employer (usually a corporation) who pays off that labor liability $-L$ as well as the liabilities for the other inputs $-K$ and gets the ownership of the product Q so in vectorial terms the employer is recognized as owning and owing the vector of assets and liabilities $\mathbf{WP} = (Q, -K, -L)$. Hence the people working in the opportunity (including managers) are responsible for producing

$$\mathbf{WP_L} = (Q, -K, 0) = (Q, -K, -L) + (0, 0, -L) = \mathbf{WP} + (0, 0, L)$$

but they only get the ownership of their labor $(0, 0, L)$. As John Bates Clark put it:

> A plan of living that should force men to leave in their employer's hands anything that by right of creation is theirs, would be an institutional robbery—a legally established violation of the principle on which property is supposed to rest. [1, p. 9]

The institutional robbery is the difference between what "by right of creation is theirs" $\mathbf{WP_L}$ and what they are recognized as owning $(0, 0, L)$, namely

$$\mathbf{WP_L} - (0, 0, L) = (Q, -K, 0) - (0, 0, L) = (Q, -K, -L) = \mathbf{WP}$$

which is the whole product $\mathbf{WP}$. And the value of the institutional robbery $\mathbf{WP}$ is the profits:

$$\mathbf{P} \cdot \mathbf{WP} = (p, r, w) \cdot (Q, -K, -L) = pQ - rK - wL.$$

These results are summarized in the following table.

[5] One could, of course, do the symmetrical integration of $\mathbf{MWP_K}$ from 0 to K to obtain $\mathbf{WP_K} = (Q, 0, -L)$ but it would not have the same significance since the services of things (represented as the capital services K) are not capable of being responsible (in the usual juridical sense) for anything.

[6] "Since slavery was abolished, human earning power is forbidden by law to be capitalized. A man is not even free to sell himself: he must *rent* himself at a wage." [11, p. 52 (his italics)]

Property imputations	Property assets and liabilities	Net value
Labor de facto responsible for whole product of labor **WP$_L$**	(Q, –K, 0)	Value-added pQ – rK
Labor legally appropriates labor commodity L	(0, 0, L)	Labor costs wL
Labor responsible for but does not appropriate the whole product **WP**	(Q, –K, 0) –(0, 0, L) = (Q, –K, –L)	Profits pQ –rK – wL

Figure 4: Property imputations in an employment firm.

Since this is obviously an unsatisfactory "scientific account" for the whole system of renting persons, the orthodox economics profession invariably presents MP theory using the scalar notion of marginal product and uses the metaphorical notion of all the causally efficacious factors of production as being "responsible" for their share of the outputs–which then seems to satisfy the metaphorical imputation principle:

> The basic postulate on which the argument rests is the ethical proposition that an individual deserves what is produced by the resources he owns. [5, p. 199]

Thus considering the alternative, the advantages of the scalar MP theory are quite clear.

8 Mathematical Appendix

8.1 Standard MP Theory

Let $y = f(x_1, ..., x_n)$ be a smooth neoclassical production function with p as the competitive unit price of the output y and $w_1, ..., w_n$ as the respective competitive unit prices of the inputs $x_1, ..., x_n$. The cost minimization problem involves the input prices and a given level of output y_0:

$$\text{minimize: } C = \sum_{i=1}^{n} w_i x_i$$
$$\text{subject to: } y_0 = f(x_1, ..., x_n)$$

Minimize Cost to Produce Given Output

Forming the Lagrangian

$$L = \sum_{i=1}^{n} w_i x_i - \lambda (y_0 - f(x_1, ..., x_n)),$$

the first-order conditions

$$\frac{\partial L}{\partial x_i} = w_i - \lambda \frac{\partial f}{\partial x_i} = 0 \text{ for } i = 1, ..., n$$

solve to:

$$\lambda = \frac{w_1}{\partial f / \partial x_1} = ... = \frac{w_n}{\partial f / \partial x_n}.$$

First-Order Conditions for Cost Minimization

These equations together with the production function determine the n unknowns $x_1, ..., x_n$. Varying the input prices and level of output parametrically determines the conditional factor demand functions:

$$x_1 = \varphi_1(w_1, ..., w_n, y)$$
$$\vdots$$
$$x_n = \varphi_n(w_1, ..., w_n, y).$$
Conditional Factor Demand Functions

These functions give the optimum level of the inputs to minimize the cost to produce the given level of output at the given input prices. Taking the input prices as fixed parameters, we can write the conditional factor demand functions as $x_i = \varphi_i(y)$ for $i = 1, ..., n$. These functions define the cost-minimizing expansion path through input space parameterized by the level of output. Substituting into the sum for total costs yields the

$$C(y) = \sum_{i=1}^{n} w_i \varphi_i(y).$$
Cost Function

Differentiation by y yields the marginal cost function.

$$MC = \frac{dC}{dy} = \sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y}.$$
Marginal Cost

The factor demand functions can also be substituted into the production function to obtain the identity:

$$y = f(\varphi_1(y), ..., \varphi_n(y)).$$

Differentiating both sides with respect to y yields the useful equation:

$$1 = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial \varphi_i}{\partial y}.$$

Multiplying both sides by the Lagrange multiplier allows us to identity λ as the marginal cost.

$$\lambda = \sum_{i=1}^{n} \left(\lambda \frac{\partial f}{\partial x_i} \right) \frac{\partial \varphi_i}{\partial y} = \sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y} = MC.$$
Lagrange Multiplier of Minimum Cost Problem is Marginal Cost

Using the customary marginal product notation, $MP_i = \partial f / \partial x_i$ for $i = 1, ..., n$, the first order conditions for cost minimization can be written as:

$$MC = \frac{w_1}{MP_1} = ... = \frac{w_n}{MP_n}.$$
Cost Minimization Conditions

The marginal products should not be confused with the reciprocals of the factor demand function partials:

$$\frac{\partial f}{\partial x_i} \neq 1 / \frac{\partial \varphi_i}{\partial y}.$$

The marginal product $MP_i = \partial f / \partial x_i$ of x_i gives the marginal increase in y when there is both a marginal increase in x_i and a shift to a more x_i-intensive production technique so that exactly the same amount of the other inputs is used. No factor prices or cost minimization is involved in the definition. The reciprocal of $\partial \varphi_i / \partial y$ gives the marginal increase in y associated with a marginal increase in x_i when there is a corresponding increase in the other inputs so as to produce the extra output at minimum cost.

8.2 MP Theory with Product Vectors

For the inclusive algebraically symmetric notion of the product, we will use vectors with the outputs listed first followed by components for the inputs. The positive product is $(y, 0, ..., 0)$, the negative product is $(0, -x_1, ..., -x_n)$, and their sum is the

$$\mathbf{WP} = (y, -x_1, ..., -x_n)$$

Whole Product Vector

The whole product vector is usually called the "production vector" or "net output vector" [12, p. 8] in the set-theoretic presentations using production sets rather than production functions. Assuming that costs are minimized at each output level, we can restrict attention to the whole product vectors along the expansion path:

$$\mathbf{WP}(y) = (y, -\varphi_1(y), ..., -\varphi_n(y)).$$

The gradient $\triangledown_y = \frac{\partial}{\partial y}$ operator applied to the whole product vector is the *marginal whole product* **MWP**.

$$\mathbf{MWP}(y) = \triangledown_y \mathbf{WP}(y) = \left(1, -\frac{\partial \varphi_1}{\partial y}, ..., -\frac{\partial \varphi_n}{\partial y}\right).$$

Marginal Whole Product Vector **MWP**

The price vector is $\mathbf{P} = (p, w_1, ..., w_n)$, the value of the whole product (the dot product of the price and whole product vectors) is the profit.

$$\mathbf{P} \cdot \mathbf{WP} = py - \sum_{i=1}^{n} w_i x_i = py - C(y),$$

Value of Whole Product = Profit

and the *value of the marginal whole product* is the

$$\mathbf{P} \cdot \mathbf{MWP}(y) = \mathbf{P} \cdot \left(1, -\frac{\partial \varphi_1}{\partial y}, ..., -\frac{\partial \varphi_n}{\partial y}\right) = p - \sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y} = p - MC.$$

Value of Marginal Whole Product = Marginal Profit

The necessary condition for profit maximization is that the marginal whole product has zero net value, which yields the familiar conditions $p = MC$. Substituting p for MC in the cost minimization conditions yields the central equations in the usual presentation of MP theory:

$$pMP_i = w_i \text{ for } i = 1, ..., n$$

which are interpreted as showing that in competitive equilibrium, each unit of a factor is paid w_i which is the value pMP_i of "what it produces" MP_i.

8.3 One Responsible Factor

We move now to the formulation of the same mathematics but with certain factors treated as responsible factors, i.e., the treatment of MP theory with responsible factors. At first we assume only one responsible factor that can be arbitrarily taken as the first factor, which provides the services x_1. In terms of totals, the responsible factor, by performing the services or actions x_1, is responsible on the positive side for producing y and is responsible on the negative side for using up the other inputs $x_2, ...x_n$. Since the customary notation lists x_1 along side the other inputs, we could also picture the responsible factor as both producing and using up x_1 (which thus cancels out). Thus the whole product of the responsible factor is:

$$\mathbf{WP}_1 = (y, 0, -x_2, ..., -x_n) = \mathbf{WP} + (0, x_1, 0, ..., 0).$$
Whole Product of Responsible Factor x_1

The whole product of the responsible factor is formally the sum of the whole product and the services of the responsible factor.

Since we are now assuming only one responsible factor, we have the luxury of mathematically treating its actions as the independent variable. Restricting attention to the expansion path as usual and assuming $\partial\varphi_1/\partial y \neq 0$, we can invert the first factor demand function to obtain

$$y = \varphi_1^{-1}(x_1)$$

which can be substituted into the other factor demand functions to obtain the other inputs as functions of x_1:

$$x_i = \varphi_i\left(\varphi_1^{-1}(x_1)\right) \text{ for } i = 2, ..., n.$$

The whole product of the responsible factor can then be expressed as a function of x_1:

$$\mathbf{WP}_1(x_1) = \left(\varphi_1^{-1}(x_1), 0, -\varphi_2\left(\varphi_1^{-1}(x_1)\right), ..., -\varphi_n\left(\varphi_1^{-1}(x_1)\right)\right).$$
Whole Product of Responsible Factor x_1 as a Function of x_1

We can now present a realistic picture of the effects of a marginal increase in the responsible factor. A marginal increase in x_1 with both use up the other factors at the rate

$$\frac{\partial\varphi_i\left(\varphi_1^{-1}\right)}{\partial x_1} = \frac{\partial\varphi_i/\partial y}{\partial\varphi_1/\partial y}$$

and will increase the output at the rate

$$\frac{\partial\varphi_1^{-1}}{\partial x_1} = \frac{1}{\partial\varphi_1/\partial y}$$

along the expansion path. This information is given by the x_1 gradient $\bigtriangledown_1 = \frac{\partial}{\partial x_1}$ of the whole product of the responsible factor, which is the marginal whole product of the responsible factor:

$$\mathbf{MWP_1} = \bigtriangledown_1 \mathbf{WP_1}(x_1) = \left(\frac{1}{\partial\varphi_1/\partial y}, 0, \frac{\partial\varphi_2/\partial y}{\partial\varphi_1/\partial y}, ..., \frac{\partial\varphi_n/\partial y}{\partial\varphi_1/\partial y}\right).$$
Marginal Whole Product of Responsible Factor

This marginal whole product vector $\mathbf{MWP_1}$ presents what the responsible factor is marginally responsible for in quantity terms. Thus it should be compared with the marginal product MP_1 in the conventional treatment of MP theory. The marginal product MP_1 is fine as a mathematically defined partial derivative. But to interpret it in terms of production, one has to consider the purely notional shift to a more x_1-intensive productive technique so that exactly the same amount of the other factors is consumed. That is *not* how output changes in the cost-minimizing firm. The marginal whole product $\mathbf{MWP_1}$ presents the marginal changes in the output and the other factors associated with a marginal increase in x_1 along the expansion path.

The value of the marginal whole product of x_1 is the dot product:

$$\mathbf{P} \cdot \mathbf{MWP_1} = \frac{[p - w_2\partial\varphi_2/\partial y - ... - w_n\partial\varphi_n/\partial y]}{\partial\varphi_1/\partial y} = \frac{[p - MC]}{\partial\varphi_1/\partial y} + w_1.$$
Value of Marginal Whole Product of Responsible Factor

Thus we have that the necessary condition for profit maximization, $p = MC$, is equivalent to

$$\mathbf{P} \cdot \mathbf{MWP_1} = w_1.$$
Profit Max Implies Value of Marginal Whole Product
= Opportunity cost of using marginal unit of responsible factor

Production is carried to the point where the value of the marginal whole product of the responsible factor is equal to its opportunity cost given by w_1. Since we are assuming cost minimization, this is also equivalent to the conventional equation: $pMP_1 = w_1$.

8.4 Example With Substitution: Cobb-Douglas Production Function

For a Cobb-Douglas production function, $Q = AK^a L^b$ with the unit prices of K and L being r and w, one can derive the vectorial marginal products by taking derivatives along the least cost expansion path. To fill in the details, the first-order conditions for cost-minimization, $\frac{w_1}{\partial f/\partial x_1} = ... = \frac{w_n}{\partial f/\partial x_n}$, are in this case:

$$\frac{r}{aAK^{a-1}L^b} = \frac{w}{bAK^a L^{b-1}} \text{ or } \frac{AK^a L^{b-1}}{AK^{a-1}L^b} = \frac{K}{L} = \frac{aw}{br}$$

so the conditions that hold along the least-cost expansion path are:

$$\frac{K}{L} = \frac{aw}{br}.$$

The whole product of the responsible factor L, previously $\mathbf{WP}_1 = (y, 0, -x_2, ..., -x_n)$, can then be stated directly as a function of L using $K = \frac{aw}{br} L$:

$$\mathbf{WP_L} = (Q(L), -K(L), 0) = \left(A\left(\frac{awL}{br}\right)^a L^b, -\frac{aw}{br}L, 0\right) = \left(A\left(\frac{aw}{br}\right)^a L^{a+b}, -\frac{aw}{br}L, 0\right).$$

Taking the derivative with respect to L gives the marginal whole product of the responsible factor labor:

$$\mathbf{MWP_L} = \left((a+b)A\left(\frac{aw}{br}\right)^a L^{a+b-1}, -\frac{aw}{br}, 0\right).$$

Since labor is the only responsible factor, one can compute its total responsibility for the positive and negative results of production by "adding up" or integrating its marginal whole product from 0 to L to obtain the result—which is the whole product of the responsible factor: $(Q, -K, 0)$:

$$\int_0^L \mathbf{MWP_L} dl = \int_0^L \left((a+b)A\left(\frac{aw}{br}\right)^a l^{a+b-1}, -\frac{aw}{br}, 0\right) dl = \left(A\left(\frac{aw}{br}\right)^a l^{a+b}\Big]_0^L, -\frac{aw}{br}l\Big]_0^L, 0\right)$$
$$= \left(A\left(\frac{aw}{br}\right)^a L^{a+b}, -\frac{aw}{br}L, 0\right) = \left(A\left(\frac{K}{L}\right)^a L^{a+b}, -\frac{K}{L}L, 0\right) = \left(AK^a L^b, -K, 0\right) = (Q, -K, 0) = \mathbf{WP_L}.$$

Orthodox MP theory metaphorically represents each factor as producing a certain share of the product Q, as if each input could produce some part of the product without using up some of the other factors. Then under the additional assumption of constant returns to scale ($a + b = 1$ in this case), it proves the "Adding Up" or "Exhaustion of the Product Theorem" that the shares add up to the entire output Q.[7] In the case at hand, the shares are:

$$KMP_K = K\frac{\partial Q}{\partial K} = KAaK^{a-1}L^b = aQ \text{ and } LMP_L = L\frac{\partial Q}{\partial L} = LbAK^a L^{b-1} = bQ$$

so the sum of the "shares" is:

$$KMP_K + LMP_L = (a+b)Q$$

which equals Q under the assumption of constant returns, $a + b = 1$.

Continuing with the Cobb-Douglas example, we will have at profit maximization both $pMP_L = w$ and $\mathbf{P} \cdot \mathbf{MWP_L} = w$ so we might compute both functions on the left hand side to compare them. The scalar MP of labor is $MP_L = bQ/L$ so as a function of L,

$$pMP_L = pbA\left(\frac{aw}{br}\right)^a L^{a+b-1}.$$

In the vectorial case, we have:

[7] See, for instance, Friedman [5, p. 194].

$$\mathbf{P}\cdot\mathbf{MWP_L} = p\,(a+b)\,A\left(\tfrac{aw}{br}\right)^{a}L^{a+b-1} - r\left(\tfrac{aw}{br}\right)$$
$$= \tfrac{a+b}{b}\left[pbA\left(\tfrac{aw}{br}\right)^{a}L^{a+b-1}\right] - \tfrac{aw}{b} = \tfrac{a+b}{b}pbA\left(\tfrac{K}{L}\right)^{a}L^{a+b-1} - \tfrac{aw}{b}$$
$$= \tfrac{a+b}{b}pMP_L - \tfrac{aw}{b} = \tfrac{a}{b}\left[pMP_L - w\right] + pMP_L.$$

Thus we finally have:

$$\mathbf{P}\cdot\mathbf{MWP_L} = \tfrac{a}{b}\left[pMP_L - w\right] + pMP_L$$

and solving for pMP_L, we have:

$$pMP_L = \tfrac{b}{a+b}\left[\mathbf{P}\cdot\mathbf{MWP_L} + \tfrac{aw}{b}\right] = \tfrac{b\mathbf{P}\cdot\mathbf{MWP_L}+aw}{a+b}$$

so the two functions are not the same. But when $pMP_L = w$, then $\mathbf{P}\cdot\mathbf{MWP_L} = w$ and when $\mathbf{P}\cdot\mathbf{MWP_L} = w$, then $pMP_L = w$, so the two conditions, one using the scalar MP_L and the other using the vectorial $\mathbf{MWP_L}$, are equivalent. This illustrates how MP theory could just as well be presented using the vectorial marginal products.

8.5 Several Jointly Responsible Factors

Continuing the mathematical treatment, the generalization to several jointly responsible factors is straightforward. The main mathematical difference is that we lose the luxury of parameterizing motion along the expansion path by "the" responsible factor, since we now assume several such factors. Hence output will be used as the independent variable to represent motion along the expansion path.

The whole product vector $\mathbf{WP(y)}$ and the marginal whole product vector $\mathbf{MWP(y)}$ are the same as before. Suppose there are m jointly responsible factors, which we can take to be the first m factors. Intuitively, by performing the services or actions $x_1, ..., x_m$, the responsible factors use up the inputs $x_{m+1}, ..., , x_n$ and produce the outputs y. As before the *whole product of the responsible factors*, now symbolized $\mathbf{WP_r(y)}$, can be presented as the sum of the whole product and the services of the responsible factors:

$$\mathbf{WP_r}\,(y) = \mathbf{WP} + (0, \varphi_1\,(y)\,, ..., \varphi_m\,(y)\,, 0, ..., 0)$$
$$= (y, 0, ..., 0, -\varphi_{m+1}\,(y)\,, ..., -\varphi_n\,(y)).$$

Whole Product of Responsible Factors $x_1, ..., x_m$

The *marginal whole product of the responsible factors* (with variation parameterized by y) is the gradient $\nabla_y = \frac{\partial}{\partial y}$ of $\mathbf{WP}_r(y)$:

$$\mathbf{MWP_r}\,(y) = \nabla_y\mathbf{WP_r}\,(y) = \left(1, 0, ..., 0, -\tfrac{\partial\varphi_{m+1}}{\partial y}, ..., \tfrac{\partial\varphi_n}{\partial y}\right)$$

Marginal Whole Product of Responsible Factors $x_1, ..., x_m$

and its value is the dot product with the price vector.

$$\mathbf{P}\cdot\mathbf{MWP_r} = p - \textstyle\sum_{j=m+1}^{n} w_j \tfrac{\partial\varphi_j}{\partial y} = \left[p - MC\right] + \textstyle\sum_{i=1}^{m} w_i \tfrac{\partial\varphi_i}{\partial y}$$

Value of Marginal Whole Product of Responsible Factors $x_1, ..., x_m$.

When producing the marginal increase in output by using up the marginal amounts of the other inputs, the responsible factors use up the marginal services

$$\left(0, \tfrac{\partial\varphi_1}{\partial y}, ..., \tfrac{\partial\varphi_m}{\partial y}, 0, ..., 0\right)$$

which have the opportunity cost of

$$\sum_{i=1}^{m} w_i \frac{\partial \varphi_i}{\partial y}.$$
Opportunity Cost of Marginal Responsible Services for Marginal Increase in y

Hence value is maximized when the responsible factors carry production to the point when the value of their marginal whole product is equal to their marginal opportunity cost which is clearly equivalent to the equation: $p = MC$.

$$\mathbf{P} \cdot \mathbf{MWP_r} = [p - MC] + \sum_{i=1}^{m} w_i \frac{\partial \varphi_i}{\partial y} = \sum_{i=1}^{m} w_i \frac{\partial \varphi_i}{\partial y} \text{ when } p = MC$$
Value of Marginal Whole Product of Responsible Services = Their Opportunity Cost.

8.6 Several Outputs

To treat a case where there is no substitutability, i.e., input-output theory where the usual marginal products are not defined, we need to first treat a case of multiple outputs. To illustrate the generalization to several outputs, we consider an example with two products y_1 and y_2. The production possibilities can be given in the form:

$$F(y_1, y_2, x_1, ..., x_n) = 0.$$

Given the output levels y_1 and y_2, the cost minimization problem is:

$$\text{minimize: } C = \sum_{i=1}^{n} w_i x_i$$
$$\text{subject to: } F(y_1, y_2, x_1, ..., x_n) = 0.$$
Cost Minimization Problem with Two Outputs

The Lagrangian is

$$L = \sum_{i=1}^{n} w_i x_i - \lambda F(y_1, y_2, x_1, ..., x_n)$$

and the first-order conditions are $w_i - \lambda \partial F / \partial x_i = 0$ for $i = 1, ..., n$. Determining the cost-minimizing input levels in terms of the given output levels (and fixed input prices) yields the conditional factor demand functions:

$$x_i = \varphi_i (y_1, y_2) \text{ for } i = 1, ..., n.$$
Conditional Factor Demand with Two Outputs

Substituting into the sum of costs yields the cost function $C(y_1, y_2)$ and the marginal costs:

$$MC_j = \frac{\partial C}{\partial y_j} = \sum_{i=1}^{n} w_i \frac{\partial \varphi_i}{\partial y_j} \text{ for } j = 1, 2.$$
Marginal Costs of the Two Outputs

The whole product vector (parameterized by y_1 and y_2) is

$$\mathbf{WP}(y_1, y_2) = (y_1, y_2, -\varphi_1 (y_1, y_2), ..., -\varphi_n (y_1, y_2))$$

and the two marginal whole products with respect to y_1 and y_2 are the two gradients $\bigtriangledown_j = \frac{\partial}{\partial y_j}$ with respect to those variables:

$$\bigtriangledown_1 \mathbf{WP} = \left(1, 0, -\frac{\partial \varphi_1}{\partial y_1}, ..., -\frac{\partial \varphi_n}{\partial y_1}\right)$$
$$\bigtriangledown_2 \mathbf{WP} = \left(0, 1, -\frac{\partial \varphi_1}{\partial y_2}, ..., -\frac{\partial \varphi_n}{\partial y_2}\right)$$
Marginal Whole Products with Respect to the Two Outputs

With output unit prices p_1 and p_2, the value of the marginal whole products must be zero for profits to be maximized:

$$\mathbf{P} \cdot \bigtriangledown_j \mathbf{WP} = (p_1, p_2, w_1, ..., w_n) \cdot \bigtriangledown_j \mathbf{WP} = p_j - MC_j = 0 \text{ for } j = 1, 2.$$

Profit Maximization Conditions for Multiple Outputs

Let the first m factors be the responsible factors as before. The whole product of the responsible factors is the sum of the whole product and the services of the responsible factors:

$$\mathbf{WP_r}(y_1, y_2) = (y_1, y_2, 0, ..., 0, -\varphi_{m+1}(y_1, y_2), ..., -\varphi_n(y_1, y_2))$$

Whole Product of Responsible Factors

and the marginal whole products of the responsible factors would be the two gradients with respect to y_1 and y_2. The values of those marginal whole products are:

$$\mathbf{P} \cdot \bigtriangledown_j \mathbf{WP_r} = [p_j - MC_j] + \sum_{i=1}^{m} w_i \frac{\partial \varphi_i}{\partial y_j} \text{ for } j = 1, 2.$$

Value of Marginal Whole Products of Responsible Factors for each of the Two Outputs

To produce a marginal increase in y_1, the responsible factors must used actions which have the marginal opportunity cost:

$$\sum_{i=1}^{m} w_i \frac{\partial \varphi_i}{\partial y_j}$$

Marginal Opportunity Cost of Responsible Factors for Marginal Increase in y_1

and similarly for y_2. Value is maximized when the responsible factors carry production of each output to the point when the value of their respective marginal whole product is equal to their respective marginal opportunity costs:

$$\mathbf{P} \cdot \bigtriangledown_j \mathbf{WP_r} = \sum_{i=1}^{m} w_i \frac{\partial \varphi_i}{\partial y_j} \text{ for } j = 1, 2.$$

Profit Max Implies Value of Marginal Whole Products of Responsible Factors Is Their Opportunity Cost

which is clearly equivalent to $p_j = MC_j$ for $j = 1, 2$. This example with several products helps to motivate the next multi-product model where there is no substitution.

8.7 Example Without Substitution: Input-Output Theory

We have criticized the usual interpretation of MP_i as the "product of the marginal unit of x_i" on a number of grounds. For instance, a marginal increase in x_i cannot produce an increase in the output out of thin air. Other inputs will be needed. The definition of the partial derivative MP_i however assumes substitutability in the sense that there is a shift to a slightly more x_i intensive productive technique so that more output can be produced using exactly the same amount of the other factors. Yet we have shown that such an imaginary shift is not necessary to interpret marginal productivity theory. By using vectorial notions of the product, MP theory can be expressed using marginal whole products computed along the cost-minimizing expansion path–even when scalar MP are not defined. The luxury of the alternative treatment of MP theory becomes a necessity when there is no substitutability as in a Leontief input-output model.

We will consider an example where there are n commodities $x_1, ..., x_n$ and labor L where the latter is taken as the services of the responsible factor. The technology is specified by the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ where a_{ij} gives the number of units of the i^{th} good needed per unit of the j^{th} good as output. Thus for the output column vector $\mathbf{x} = (x_1, ..., x_n)^T$ (the superscript "T" denotes the transpose), the vector of required commodity inputs is $\mathbf{Ax}$. The labor requirements per unit are given by the vector $\mathbf{a}_0 = (a_{01}, ..., a_{0n})$, so the total labor requirement is the scalar $L = \mathbf{a}_0\mathbf{x}$.

Let $\mathbf{p} = (p_1, ..., p_n)$ be the price vector and let w be the wage rate or opportunity cost of a unit of labor. We assume that the outputs and inputs are separated by one time period (a "year") and that r is the annual interest rate. The competitive equilibrium condition is usually stated as the zero-profits condition with no mention of marginal productivity or the like. With labor taking its income at the end of the year, the zero-profit condition for any output vector is:

$$\mathbf{px} = (1+r)\mathbf{pAx} + w\mathbf{a_0x}.$$

Since this must hold for any $\mathbf{x}$, we can extract the following vector equation.

$$\mathbf{p} = (1+r)\mathbf{pA} + w\mathbf{a_0}$$

Competitive Equilibrium Condition

We now show how this condition can be derived using MP-style reasoning with products represented as vectors. The whole product will be a $2n+1$ component column vector since the output vector $\mathbf{x}$ is produced a year after the input vector $\mathbf{Ax}$. The following notation for the whole product is self-explanatory:

$$\mathbf{WP} = \begin{bmatrix} \mathbf{x} \\ -\mathbf{Ax} \\ -\mathbf{a}_0\mathbf{x} \end{bmatrix}$$

Whole Product Vector **WP**

The whole product of the responsible factor, labor, is the sum of the whole product and the services of the responsible factor (since the factor is represented as both producing and using up its own services):

$$\mathbf{WP_L} = \mathbf{WP} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{a}_0\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ -\mathbf{Ax} \\ 0 \end{bmatrix}$$

Whole Product of Labor Vector $\mathbf{WP_L}$

To consider output variations, we use the output unit vectors $\delta_j = (0, ..., 0, 1, 0, ..., 0)^T$ where the "1" is in the j^{th} place. The marginal whole product of the responsible factor with respect to the j^{th} output is will be symbolized as:

$$\bigtriangledown_j \mathbf{WP}_L = \begin{bmatrix} \delta_j \\ -\mathbf{A}\delta_j \\ 0 \end{bmatrix}$$

Marginal Whole Product of Labor with Respect to the j^{th} Output

and the required labor is $\mathbf{a}_0\delta_j = a_{0j}$ with the opportunity cost of wa_{0j}. The price vector stated in year-end values is $\mathbf{P} = (\mathbf{p}, (1+r)\mathbf{p}, w)$ so the value of the marginal whole product of labor is:

$$\mathbf{P} \cdot \bigtriangledown_j \mathbf{WP_L} = p_j - (1+r)\,\mathbf{pA}\delta_j$$

Value of Marginal Whole Product of Labor with respect to the j^{th} Output for $j = 1, ..., n$.

When the value of that marginal whole product of the responsible factor with respect to the j^{th} output is set equal to opportunity cost of the necessary labor wa_{0j} for $j = 1, ..., n$, then we again have the same equilibrium conditions:

$$\mathbf{p} - (1+r)\mathbf{pA} = w\mathbf{a_0}.$$

Competitive Equilibrium Condition Expressed as:
Value of Marginal Whole Product of Labor with Respect to Each Output = Its Opportunity Cost.

Thus the alternative vectorial presentation of MP theory is not only equivalent where scalar MPs can be defined but also can be used in models without substitution where the scalar MPs are undefined.

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