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# Double Entry Multidimensional Accounting

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**This paper presents a model of double entry multidimensional accounting in 'physical terms' using vectors of property rights. Property accounting gives a valuation-free description of the property transactions underlying the value transactions of ordinary accounting. Thus it avoids the valuation controversies of value accounting. Given any vector of valuation coefficients (e.g. prices or costs), a system of value accounting can be derived from a valuation-free system of property accounting by multiplying the property vectors by the value vector. The extension of double entry accounting to vectors is based on the modern mathematical formulaton of double entry bookkeeping using the group of differences.**

## INTRODUCTION

THIS ARTICLE presents a system of *property accounting*. It is the first complete and valuation-free system of accounting. The valuation controversies of ordinary accounting are undercut by the property accounting system which keeps accounts directly in terms of vectors of the underlying property rights themselves—instead of using some scalar measure of the value of those rights.

The mathematical framework of property accounting, namely *vector accounting*, is based on the modern mathematical treatment of double entry bookkeeping using the group of differences or 'Pacioli group' [10]. There is, in modern algebra, a standard construction called the *group of differences*. In a university modern algebra course, it is used to construct the (positive and negative) integers using ordered pairs of natural numbers (non-negative integers). The mathematical treatment of double entry bookkeeping given here is based on the observation that the intuitive algebra of *T*-accounts used in double entry bookkeeping is precisely equivalent to that group of differences construction—which is thus renamed the 'Pacioli group'. The *T*-accounts of double entry bookkeeping are the ordered pairs of the group of differences construction.

## PREVIOUS WORK ON DOUBLE ENTRY MULTIDIMENSIONAL ACCOUNTING

Neither the mathematical treatment of double entry bookkeeping using the Pacioli group nor double entry multidimensional property accounting have previously appeared in the mathematical accounting literature (see bibliography). Since some researchers believe that double entry multidimensional property accounting has been successfully treated in the literature, we must briefly review some of the previous work.

The presentation of the transactions in value accounting can be facilitated by using a square array or table usually called a 'transactions matrix'. These transactions tables were first used by the English mathematician, Augustus DeMorgan [7], and have been popularized a century later by the American mathematician John Kemeny and his colleagues in an influential text [20].

Transactions tables have, however, retarded the development of a mathematical formulation of double entry bookkeeping. Double entry bookkeeping lives in group theory, *not* in matrix algebra. Transactions tables do not themselves constitute a complete mathematical treatment of double entry bookkeeping. The crucial accounting operations, such as finding the balance in a $T$-account, are still performed informally, i.e. by the informal comparison of a row sum and a column sum. The mathematical formulation of double entry bookkeeping using the Pacioli group allows it to be generalized to new systems of accounting such as vector and property accounting. The matrix treatment of scalar accounting exploits certain particular features of scalar accounting (i.e. in the terminology defined below, the feature is that all scalar $T$-accounts in reduced form have pure balances). These features do not extend to vector accounting so the matrix approach does not generalize to these new domains [see 9, Chap. 12, section 2, "Transactions Matrices"]. Hence transactions tables are not used here.

The most fundamental characteristic of a double entry accounting system is that it updates *an equation*, and that it uses $T$-accounts with debit and credit entries. Conventional value accounting records economic events as transactions to update the balance sheet equation. Indeed, it is the equational aspect of an accounting system which requires that two or more entries be made to update the equation. Given any equation

$$U + V + \ldots + W = X + Y + \ldots + Z$$

there is no way that only one term, such as $V$, can be changed and still maintain the truth of the equation. Two or more terms must be changed.

The systems of 'double entry multidimensional accounting' previously presented in the literature lack this most basic characteristic of an accounting system, the equational aspect (e.g. [4, 5, 12].

> "For instance, the convenient idea of an accounting identity is lost since the dimensional and metric comparability it assumes is no longer present except under special circumstances". [15, p. 333]

However, vector accounting shows that an accounting equation can still be used in the presence of incomparability between dimensions by using *vector equations*. This is a mathematical fact independent of the content. The content of the vector accounting formalism could be property accounting, social accounting, accounting for physical inventories at different locations, and so forth.

Professor Yuji Ijiri has made the most determined attempt to develop a system of double entry multidimensional accounting [13–17]. Ijiri clearly states the *idea* of a property accounting system that he calls *multi-dimensional physical accounting* [14, p. 155]. His worked-out model of double entry multi-dimensional physical accounting is presented in three separate places [14–16].

Ijiri's work is, however, bound to be fragmentary and unsuccessful without the mathematical prerequisite of double entry vector accounting using the Pacioli group. The balance sheet is presented simply as a *set* of physically incommensurate quantities (debts as negative assets). There is *no balance sheet equation* in the model. Ijiri introduces permanent $T$-accounts called 'asset accounts' and temporary $T$-accounts called 'activity accounts'. But none of the $T$-accounts use vectors; they are all scalar accounts.

Ijiri describes a number of sample physical transactions and each is carefully recorded with equal debits and credits. The debit and credit entries for different transactions are made in incommensurate quantities. This creates no problem in the asset accounts since each asset account deals only with one type of physical quantity. But the incommensurate quantities get jumbled together in the activity accounts. This means that no meaningful accounting operations can be performed on the activity accounts. In particular, the activity accounts cannot be summed and cannot be closed. All Ijiri can do is simply abandon them as a jumble of incommensurate quantities and rule them as if they had been closed.

There is no proprietorship account to close the temporary accounts into because there is no balance sheet equation. The balance sheet just gives the incommensurate physical totals for the assets and debts with no equity accounts. The abandonment of the activity or temporary accounts also raises the question of in what sense the model is 'double entry'. 'Double entry' bookkeeping makes little sense in a context where there is no equation to be updated. It is as if the second entry was made on a scrap of paper (the activity accounts) which is then simply discarded.

Ijiri rigorously recorded each transaction with an equal debit and credit entry to illustrate the trial balance in the context of multidimensional accounting.

> "Since every number is entered twice, once on the debit side and once on the credit side, the flash total of entries

on the debit side of all accounts is equal to the flash total of entries on credit side of all accounts." [14, p. 158; with a similar statement in 15, p. 113]

Firstly, this trial balance (unlike the trial balance in vector accounting) adds incommensurates together. For instance, on the debit side of the Sales account, the trial balance would be adding 3,500 cases of finished goods and 600 man-hours together to get 4,100 'whatevers'. Secondly, the trial balance cannot possibly work. It is well understood in accounting that the trial balance works because one starts *with an equation* which gives equal debits and credits in the beginning ledger. Adding on the equal debits and credits from the transactions will yield the equal debits and credits in the ending ledger. But since Ijiri does not begin with an equation, his totals are foredoomed to be unequal. He adds equals to unequals and, of course, gets unequals as a result. When one adds up the debits and the credits in his ledger [14, pp. 156–157; 15, pp. 112–113; 16, pp. 101–102], one finds that there are 12,500 more 'whatevers' on the credit side of the ledger than on the debit side.

This audit of Professor Ijiri's model of double entry multidimensional physical accounting serves to acknowledge his formulation of idea of a physical accounting system and to point out the reasons for the failure of his particular model. It is hard to understand a successful model if one fails to see how earlier models failed. Yet in the almost two decades since Ijiri's model was first published, there seems to have been no analysis and criticism of the model in the accounting journals. No one seems to have publicly pointed out such basic matters as:

(1) the lack of a balance sheet equation,

(2) the lack of any equity accounts,

(3) the nonclosure and unworkability of the temporary accounts, and

(4) the nonbalancing trial balance.

But it is not necessary to accept any of these sacrifices in the step to multidimensional accounting. All those desirable features of ordinary accounting are maintained and generalized in the model of double entry multidimensional accounting presented here.

## MONOIDS AND GROUPS

A binary operation, $x + y$, on a set $M$ is *associative* if for any $x$, $y$, and $z$ in $M$,

$$(x + y) + z = x + (y + z).$$

A set with an associative binary operation on it is a *semigroup*. An *identity* element (for the binary operation $+$) is an element $e$ such that for all elements $x, x + e = e + x = x$. An identity element is unique since if $e'$ is another such identity, then $e = e + e' = e'$. A semigroup with an identity element is a *monoid*. If the binary operation in a monoid is written as addition, then the identity element is written as the zero 0. The *inverse* of element $x$ in a monoid $M$ is an element $-x$ such that $x + (-x) = 0 = -x + x$. A *group* $G$ is a monoid in which every element has an inverse.

A semigroup, monoid, or group is *commutative* (or Abelian) if for any $x$ and $y$, $x + y = y + x$. Double entry bookkeeping uses the device of $T$-accounts to construct a (commutative) group from a (commutative) monoid. In modern mathematics, this is called the *group of differences* [e.g. 1, p. 20], but since the special case of double entry bookkeeping was first published by the Italian mathematician Luca Pacioli in 1494 [11], it should be called the *Pacioli group* [9, pp. 177–178]. Especially accessible treatments of special cases have been given by Dubisch [8, p. 17], Jacobson [19, p. 10], and MacLane and Birkhoff [21, p. 44].

## THE PACIOLI GROUP

A monoid $M$ (always commutative) is *cancellative* if for any $x$, $y$, and $a$ in $M$

$$x + a = y + a \text{ implies } x = y.$$

The Pacioli group $P(M)$ is constructed by defining an equivalence relation on the set of ordered pairs of elements from a cancellative monoid $M$. The ordered pairs (or, to be specific, equivalence classes of ordered pairs) will be represented as a generalization of a $T$-account called a *T-term* and denoted $[d//c]$ for any $d$ and $c$ in $M$. The left-hand side (LHS) entry $d$ is the *debit* entry and the right-hand side (RHS) entry $c$ is the *credit* entry. The double slash notation $[d//c]$ was suggested by Pacioli himself.

"At the beginning of each entry, we always provide "per", because, first, the debtor must be given, and immediately after the creditor, the one separated from the other by two little slanting parallels (virgolette), thus, //, . . . ." [25, p. 43]

$T$-terms add together by adding debits to debits and credits to credits:

$$[w//x] + [y//z] = [w + y//x + z].$$

The identity element is the zero $T$-term $[0//0]$. Given two $T$-terms $[w//x]$ and $[y//z]$, the *cross-sums* are the two elements of $M$ obtained by adding the credit in one $T$-term to the debit in the other. The equivalence relation between $T$-terms is defined by setting two $T$-terms *equal* if their cross-sums are equal:

$$[w//x] = [y//z] \text{ if and only if } x + y = w + z.$$

The cancellation law insures that if $[w//x] = [y//z]$ and $[y//z] = [u//v]$, then $[w//x] = [u//v]$:

$$\begin{array}{c} x + y = w + z \\ z + u = y + v \\ \hline x + u + (y + z) = w + v + (y + z). \end{array}$$

Cancelling $y + z$ yields $[w//x] = [u//v]$. If $[w//x] = [y//z]$ but $w$ does not equal $y$ and $x$ does not equal $z$, then $[w//x]$ and $[y//z]$ are said to be different *representations* of the same $T$-term. The *inverse* or negative of a $T$-term is obtained by reversing the debit and credit entries, i.e. $-[w//x] = [x//w]$, and

$$-[w//x] + [w//x] = [x//w] + [w//x] = [x + w//x + w] = [0//0].$$

That completes the construction of the Pacioli group $P(M)$ of a cancellative commutative monoid $M$.

## THE DOUBLE ENTRY METHOD

Double entry bookkeeping uses the Pacioli group $P(M)$ of a cancellative commutative monoid $M$ to perform additive and subtractive operations on an equation in $M$ (e.g. the balance sheet equation).

A $T$-term $[b//a] = [0//0]$ equal to the zero $T$-term is called a *zero-term*. The double entry method operates by *encoding* or translating equations in $M$ as zero-terms in the Pacioli group $P(M)$. Given an equation LHS = RHS in $M$, the corresponding *equational zero-term* in $P(M)$ is obtained as follows. A LHS term $d$ encodes as a *debit-balance* $T$-account $[d//0]$, and a RHS term $c$ encodes as a *credit-balance* $T$-account $[0//c]$. The sum of the resulting $T$-terms is the equational zero-term representing the equation. For example, the equation

$$1500 = 1000 + 500$$

in the additive monoid of natural numbers encodes as the equational zero-term

$$[1500//0] + [0//1000] + [0//500].$$

Since there are only plus signs between the $T$-terms in the equational zero-term, the plus signs can be left implicit to obtain the set of $T$-terms (which sum to $[0//0]$) encoding the accounts which is called the *ledger*. Thus the ledger of $T$-accounts is just the equational zero-term with the plus signs between the $T$-terms left implicit.

A valid algebraic operation on an equation will transform the equation into another equation. But each equation in $M$ encodes as a zero-term in the Pacioli group. Hence an algebraic operation which transforms equations into equations would itself encode as an algebraic operation which transforms zero-terms into zero-terms. There is one and only one such group operation: add zero. Zero plus zero equals zero. A zero-term plus a zero-term equals a zero-term. Thus any additive or subtractive operation on the original equation is expressed in double entry bookkeeping as adding a zero-term to the original equational zero-term to obtain another equational zero-term. In accounting, these operations, which change the amounts in the accounts, are called *transactions*. Hence the added-on zero-terms, which encode transactions, will be called *transactional zero-terms*.

Beginning equational zero-term
+ First transactional zero-term
+ Second transactional zero-term
+ . . .
+ Last transactional zero-term
----------------------------
= Ending equational zero-term

In conventional accounting, the list of transactional zero-terms is called the *journal*, and each transactional zero-term is expressed as the *journal entry* for the transaction. The operation of adding the transactional zero-terms to the equational zero-term is called *posting* the journal to the ledger. Hence the above sum has the form:

Beginning Ledger
+ Journal
-------
= Ending Ledger.

Since a transactional zero-term is equal to [0//0], the sum of the debit entries must equal the sum of the credit entries in the transactional zero-term. That is the familiar *double entry principle* that the debits must equal the credits in each transaction. Similarly, in an equational zero-term, the sum of the debits must also equal the sum of the credits, and that is the familiar *trial balance*.

Given the ending equational zero-term in $P(M)$, each debit-balance account $[d//c]$ would be decoded as $d - c$ and placed on the LHS of the ending equation. Any credit-balance account $[d//c]$ is decoded as $c - d$ and placed on the RHS of the ending equation.

## VECTOR ACCOUNTING

Conventional value accounting is fraught with controversy, e.g. the current vs historical cost question or the questions about the proper recognition of revenues and expenses. One might wonder if there couldn't be an accounting system which would undercut these valuation controversies and simply describe the objective underlying realities (e.g. in physical terms). For instance, regardless of whether a raw material inventory is valued using LIFO, FIFO, current entry price, or whatever, it is an objective fact that so many units of a certain type of raw material are purchased, that a certain amount is used in production, and so forth.

A double entry accounting system has been developed [9] which is the first complete valuation-free system of accounting. It is called *property* accounting because it keeps accounts directly in terms of the stocks and flows of the underlying property rights. Property accounting does not solve the valuation controversies of accounting. It attempts to stake out the objective territory that involves no valuation. With a common area of agreement clearly specified, it will perhaps be easier to delineate the areas of disagreement and the real issues at the level of valuation.

From the mathematical viewpoint, property accounting uses the machinery of vector accounting. An **n**-*dimensional vector* is (for our purposes) an ordered **n**-tuple $\mathbf{X} = (\mathbf{x}_1. \ldots ,\mathbf{x}_n)$ of scalars. Vectors of the same dimension add together by adding the corresponding components. For $\mathbf{X} = (\mathbf{x}_1, \ldots , \mathbf{x}_n)$ and $\mathbf{Y} = (\mathbf{y}_1, \ldots ,\mathbf{y}_n)$,

$$\begin{aligned}\mathbf{X} + \mathbf{Y} &= (\mathbf{x}_1, \ldots ,\mathbf{x}_n) + \mathbf{y}_1, \ldots ,\mathbf{y}_n) \\ &= (\mathbf{x}_1 + \mathbf{y}_1, \ldots ,\mathbf{x}_n + \mathbf{y}_n).\end{aligned}$$

For vector accounting, the commutative monoid $M$ can be the monoid of **n**-dimensional vectors of non-negative integers. The non-negative reals could also be used instead of the non-negative integers. One-dimensional vectors of non-negative integers can be identified with the non-negative integers themselves (natural numbers), so the usual case of value accounting with scalars is the special case of vector accounting with $\mathbf{n} = 1$.

The *minimum* of two vectors $\mathrm{Min}(\mathbf{X}, \mathbf{Y})$ is formed by taking the component-wise minimum:

$$\mathrm{Min}(\mathbf{X}, \mathbf{Y})_\mathbf{k} = \text{minimum of } \mathbf{x}_\mathbf{k} \text{ and } \mathbf{y}_\mathbf{k} \text{ for } \mathbf{k} = 1, \ldots ,\mathbf{n}.$$

In conventional value accounting, there is the operation of finding the balance in a **T**-account. For example, the balance of [3//5] is [0//2]. This generalizes in vector accounting. Each vector **T**-term $[\mathbf{X}//\mathbf{Y}]$ has a *reduced representation* $[\mathbf{X}\text{-Min}(\mathbf{X}, \mathbf{Y})//\mathbf{Y}\text{-Min}(\mathbf{X}, \mathbf{Y})]$. For instance, given the **T**-term [(3,7,2)//(5,4,6)], the minimum of the debit and credit entries is (3,4,2) and the reduced representation is

$$[(3,7,2) - (3,4,2)//(5,4,6) - (3,4,2)] = [(0,3,0)//(2,0,4)].$$

A **T**-term is said to be in *reduced form* if it is its own reduced representation (i.e. if the minimum of its debit and credit entries is the zero vector).

A **T**-term has a *pure balance* if it has the zero vector as either its debit or credit entry. Otherwise it has a *mixed balance*. In one dimension (i.e. scalar accounting), all **T**-terms in reduced form have pure balances. In larger dimensions, **T**-terms will in general have mixed balances even in reduced form, i.e. non-zero balances on both the debit and credit sides. In the above example, the vector **T**-term [(0,3,0)//(2,0,4)] is in reduced form, but it does not have purely a debit balance or a credit balance; it has a mixed balance. Each component will, however, be zero on either the debit or credit side in the reduced representation, so a reduced **T**-term in vector accounting has component-wise pure balances.

The *scalar product* of two **n**-dimensional vectors is the scalar obtained by multiplying each component of one vector times the corresponding component of the other vector and summing the resulting products. Given $\mathbf{P} = (\mathbf{p}_1, \ldots \mathbf{p}_n)$

and $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, their scalar product is

$$\mathbf{PX} = \mathbf{p}_1 x_1 + \ldots + \mathbf{p}_n x_n.$$

In the economic interpretation, **X** is a *quantity vector* where $\mathbf{x}_k$ is the number of units of the **k**th type of commodity (or property right) and **P** is a *price vector* where $\mathbf{p}_k$ is the unit price of the **k**th commodity. Then the scalar product **PX** is the *value* of the quantity vector **X** when evaluated at the price vector **P**.

Given a vector **T**-term [**X**//**Y**] and a price or cost vector **P**, the scalar product is the scalar **T**-term [**PX**//**PY**]. Given a valuation vector **P**, a system of value accounting can be *derived* from a system of property accounting. The value accounting system is obtained by multiplying each **T**-term in the property accounting system by a given vector of prices or other valuation coefficients **P** [e.g. 9, Chap. 10]. In the remaining sections, a property accounting example will be sketched (without deriving any value accounting systems).

## INTRODUCTORY PROPERTY THEORY

Double entry property accounting is the development of *property theory* within the formal framework of double entry vector accounting. Hence we must briefly review some of the basic concepts of the theory of property. Property changes by:

(1) *transactions* between legal parties, and

(2) *appropriations* (or, metaphorically, transactions with Nature).

Transactions between legal parties can be divided into two types;

(1.a) *Market transactions*, where there is an equal quid pro quo in market value, and

(1.b) *non-reciprocal transfers* between legal parties such as dividends, taxes, and gifts.

It is convenient to further subdivide market transactions into;

(1.a.1) purchases, and

(1.a.2) sales.

In a transaction, a party acquires a property right by transfer from another party or gives up a property right by transfer to another party or both. In an appropriation, a party acquires a property right but not by transfer from another party, or a party gives up a property right but not by transfer to another party or both. Since there is no other legal party involved in an appropriation, it is sometimes metaphorically considered to be a 'trade with Nature'. For example, production is often viewed as an exchange with Nature where the inputs are the property given up to Nature and the outputs are the property acquired back from Nature. This metaphor may be helpful for illustrative purposes, but 'Nature' will not be awarded by an account in property accounting.

We have briefly catalogued the ways in which 'property changes'. Any of these changes in property can be interpreted in either of two fundamental senses;

(A) as a change in the legal property rights, i.e. as a *de jure* or *legal* change, and

(B) as a change in the possession of the property, i.e. as a *de facto* or *factual* change.

For instance, in a market transaction, a legally valid contract constitutes the exchange of legal property rights. The actual delivery of the goods and the payment of the consideration constitute the factual exchange of property. The factual transfers are said to *fulfil* the contract. Both the legal transaction and the factual transaction must be recorded in property accounting. For a current cash market exchange, both the legal and factual transaction could be recorded with one accounting transaction.

For credit transactions, the legal and factual transfers are accounted for separately. In a credit purchase of inputs, there is the legal transaction wherein a present right to certain inputs is exchanged for the right to certain future-dated cash. Only the one side of the factual transaction can occur at the time of the credit transaction, namely the delivery of the purchased inputs. The other side of the factual transaction, the payment of the future-dated cash, must await that future due date. In the mean time, the unfulfilled legal transfer sits on the balance sheet as a liability.

Table 1.

| |
|---|
| Assets Bank Debt Suppliers Total A & L |
| (2500,10,6) = (1000,0,0) + (0,0,0) + (1500,10,6) |
| Whole Prod. Purchases Sales NRT dA & L |
| + (0,0,0) + (0,0,0) + (0,0,0) + (0,0,0) + (0,0,0). |

## A PROPERTY ACCOUNTING EXAMPLE

Consider a simple manufacturing enterprise which only utilizes three types of commodities, cash, outputs, and inputs, so the vectors will be three dimensional. There are no fixed assets. The cost of time and taxes will be ignored. The initial balance sheet has cash, output inventory, and input inventory as assets. The property vectors have three components: (Cash, Outputs, Inputs). The initial *Assets* vector is (2500,10,6) so there is \$2500 of cash on hand, the output inventory contains ten physical units of outputs, and the input inventory contains six physical units of the inputs.

Debts are legal obligations for future-dated cash (or other asset) payments. We will represent debts in terms of the present cash which would pay off the debt, i.e. the present value of the debt payments. Debts are owed to other legal parties, so the vector representing that debt can be labeled with the name of that party. We assume that the firm owes a bank a debt with the present value of \$1000, so it would be represented by the vector (1000,0,0). The remaining vector which completes the balance sheet identity which will be called the *Total Assets and Liabilities* vector or just the *Total A & L* vector. The initial balance sheet vector equation is:

| Assets | | Bank Debt | | Total A & L |
|---|---|---|---|---|
| (2500,10,6) | = | (1000,0,0) | + | (1500,10,6). |

The Total A & L property account records the total legal rights and obligations of the legal party. Temporary or flow accounts will be associated with it to record changes in legal rights and obligations. The summary flow account associated with the Total A & L will simply be called *Changes in A & L* or simply *d A & L*. This summary flow account will be subdivided into other flow accounts which record the various specific ways that property rights change such as market transactions, non-reciprocal transfers, and appropriations. The market transactions will be recorded in two accounts, *Purchases* and *Sales*. The non-reciprocal transfers account will be abbreviated *NRT*. The appropriations will be recorded in the *Whole Product* account.

Debts owed by the firm and debts owed to the firm require personal accounts for the creditors (such as the Bank Debt account) and debtors. Since credit transactions create such debts, we require a personal account for each party involved in a credit transaction. In the economic activity being modeled in the example, we will assume a credit purchase of some inputs from suppliers so there will be a Suppliers account. With the flow accounts and the Suppliers account added in (all with zero balances), the initial balance sheet vector equation is shown in Table 1.

This initial equation is then encoded as the initial equational zero-term, (shown in Table 2). Since the $T$-accounts can only be added together, we can leave the plus signs implicit so that we have the set of property $T$-accounts, namely, the *property ledger*.

The following is the list of the economic events which we assume to take place:

1. 15 units of the inputs are purchased and delivered for \$5 cash each.
2. Contract for credit purchase of 5 units of inputs at \$5 each.
3. Suppliers deliver the 5 units purchased on credit.
4. 18 units of the inputs are used-up in production.

Table 2.

| | | |
|---|---|---|
| Assets | Bank Debt | Suppliers |
| [(2500,10,6)//(0,0,0)] + | [(0,0,0)//(1000,0,0)] + | [(0,0,0)//(0,0,0)] |
| Total A & L | Whole Product | Purchases |
| + [(0,0,0)//(1500,10,6)] + | [(0,0,0)//(0,0,0)] + | [(0,0,0)//(0,0,0)] |
| Sales | NRT | dA & L |
| + [(0,0,0)//(0,0,0)] + | [(0,0,0)//(0,0,0)] + | [(0,0,0)//(0,0,0)]. |

Table 3.

| Transaction Number | Accounts and Description | [Debit//Credit] |
|---|---|---|
| | Assets | [(0,0,15)//(75,0,0)] |
| 1 | Purchases | [(75,0,0)//(0,0,15)] |
| | Cash purchase of inputs | |
| | Suppliers | [(0,0,5)//(25,0,0)] |
| 2 | Purchases | [(25,0,0)//(0,0,5)] |
| | Contract to purchase inputs on credit | |
| | Assets | [(0,0,5)//(0,0,0)] |
| 3 | Suppliers | [(0,0,0)//(0,0,5)] |
| | Delivery of purchased inputs | |
| | Whole Product | [0,0,18)//(0,0,0)] |
| 4 | Assets | [(0,0,0)//(0,0,18)] |
| | Inputs used-up in production | |
| | Assets | [(0,36,0)//(0,0,0)] |
| 5 | Whole Product | [(0,0,0)//(0,36,0)] |
| | Outputs produced in production | |
| | Assets | [(120,0,0)//(0,40,0)] |
| 6 | Sales | [(0,40,0)//(120,0,0)] |
| | Cash sale of outputs | |
| | Bank Debt | [(200,0,0)//(0,0,0)] |
| 7 | Assets | [(0,0,0)//(200,0,0)] |
| | Principal payment on bank debt | |
| | NRT | [(100,0,0)//(0,0,0)] |
| 8 | Assets | [(0,0,0)//(100,0,0)] |
| | Payment of dividends | |

5. 36 units of the outputs are produced.

6. 40 units of the outputs are sold and delivered for $3 cash each.

7. A $200 principal payment is made on the bank debt.

8. A $100 dividend is paid.

Each event can be encoded as a transaction zero-term. The list of the transaction zero-terms with the affected $T$-accounts is the *property journal.* (See Table 3).

The temporary or flow accounts, Whole Product, Purchases, Sales, and NRT are then closed into the summary flow account dA & L, and then it is closed into Total A & L. (See Table 4).

The list of the property $T$-accounts in the equation zero-term is the *property-ledger.* (See Table 5).

Dropping the closed flow accounts, we can reinsert the plus signs between the permanent or stock property $T$-accounts to obtain the ending equational zero-term. (See Table 6).

Each $T$-account can then be decoded according to its side in the original balance sheet

Table 4.

| Transaction Number | Accounts and Description | [Debit//Credit] |
|---|---|---|
| | Whole Product | [(0,36,0)//(0,0,18)] |
| C1 | dA & L | [(0,0,18)//(0,36,0)] |
| | Close Whole Product into dA & L | |
| | Purchases | [(0,0,20)//(100,0,0)] |
| C2 | dA & L | [(100,0,0)//(0,0,20)] |
| | Close Purchases into dA & L | |
| | Sales | [(120,0,0)//(0,40,0)] |
| C3 | dA & L | [(0,40,0)//(120,0,0)] |
| | Close Sales into dA & L | |
| | NRT | [(0,0,0)//(100,0,0)] |
| C4 | dA & L | [(100,0,0//(0,0,0)] |
| | Close NRT into dA & L | |
| | dA & L | [(0,0,2)//(80,4,0)] |
| C5 | Total A & L | [(80,4,0)//(0,0,2)] |
| | Close dA & L into Total A & L | |

Table 5.

| Assets | Bank Debt |
|---|---|
| [(2500,10,6)//(0,0,0)] | [(0,0,0)//(1000,0,0)] |
| (1) [(0,0,15)//(75,0,0)] | (7) [(200,0,0)//(0,0,0)] |
| (3) [(0,0,5)//(0,0,0)] | |
| (4) [(0,0,0)//(0,0,18)] | [(200,0,0)//(1000,0,0)] |
| (5) [(0,36,0)//(0,0,0)] | = [(0,0,0)//(800,0,0)] |
| (6) [(120,0,0)//(0,40,0)] | |
| (7) [(0,0,0//(200,0,0)] | |
| (8) [(0,0,0//(100,0,0)] | |
| [(2620,46,26//(375,40,18)] | |
| = [(2245,6,8)//(0,0,0)] | |
| Suppliers | Total A & L |
| (2) [(0,0,5)//(25,0,0)] | [(0,0,0)//(1500,10,6)] |
| (3) [(0,0,0)//(0,0,5)] | (C5) [(80,4,0)//(0,0,2)] |
| [(0,0,5)//(25,0,5)] | [(80,4,0)//(1500,10,8)] |
| = [(0,0,0)//(25,0,0)] | = [(0,0,0)//(1420,6,8)] |
| Whole Product | Purchases |
| (4) [(0,0,18)//(0,0,0)] | (1) [(75,0,0)//(0,0,15)] |
| (5) [(0,0,0//(0,36,0)] | (2) [(25,0,0)//(0,0,5)] |
| (C1) [(0,36,0)//(0,0,18)] | (C2) [(0,0,20)//(100,0,0)] |
| Sales | NRT |
| (6) [(0,40,0)//(120,0,0)] | (8) [(100,0,0)//(0,0,0)] |
| (C3) [(120,0,0)//(0,40,0)] | (C4) [(0,0,0)//(100,0,0)] |
| dA & L | |
| (C1) [(0,0,18)//(0,36,0)] | |
| (C2) [(100,0,0)//(0,0,20)] | |
| (C3) [(0,40,0)//(120,0,0)] | |
| (C4) [(100,0,0)//(0,0,0)] | |
| [(200,40,18)//(120,36,20)] | |
| = [(80,4,0)//(0,0,2)] | |
| (C5) [(0,0,2)//(80,4,0)] | |

equation. This yields the *Final balance sheet vector equation:*

| Assets | | Bank Debt | | Suppliers | | Total A &L |
|---|---|---|---|---|---|---|
| (2245,6,8) | = | (800,0,0) | + | (25,0,0) | + | (1420,6,8) . |

The income statement in value accounting could be defined as the statement which connects the Net Worth accounts in the beginning and ending balance sheet equations. In property accounting, the corresponding statement would be the *property flow statement* which connects the Total A & L accounts in the beginning and ending balance sheet vector equations. Since all those changes are channeled through the summary flow account, dA & L, the property flow statement is just a list of the activity in the dA & L account.

Property Flow Statement

Whole Product = [(0,0,18)//(0,36,0)]
Purchases = [(100,0,0)//(0,0,20)]
Sales = [(0,40,0)//(120,0,0)]
NRT = [(100,0,0)//(0,0,0)]
-----------------
dA &L = [(200,40,18)//(120,36,20)]
= [(80,4,0)//(0,0,2)]

The dA & L account connects together the beginning and ending Total A & L accounts in the sense that:

| Beginning Total A & L | | dA & L |
|---|---|---|
| [(0,0,0)//(1500,10,6)] | + | [(80,4,0)//(0,0,2)] |

Ending Total A & L
= [(0,0,0)//(1420,6,8)]

The closing balance in the Whole Product account shows that the liabilities for 18 units of

Table 6.

| Assets | Bank Debt |
|---|---|
| [(2245,6,8)//(0,0,0)] + | [(0,0,0)//(800,0,0)] |
| Suppliers | Total A & L |
| + [(0,0,0)//(25,0,0)] + | [(0,0,0)//(1420,6,8)] |

inputs were appropriated in production (i.e. 18 units of the inputs were the assets expropriated in production) and that 36 units of outputs were the assets appropriated in production. The Whole Product $T$-account is an RHS account so it decodes as the *whole product vector*, $(0,36,-18)$. Whole product vectors are used in economics (without the cash component) in the modern production set representation of production opportunities where they are variously called *production vectors*, *activity vectors*, or *input-output vectors* [e.g. 26, p. 27].

A system of value accounting (balance sheet equation, journal, and ledger) can be derived from the above (highly simplified) property accounting system by multiplying each property vector by a vector of valuation coefficients such as prices or costs. Different rules for defining costs or recognizing revenue would lead to different value vectors and different derived systems of value accounting. But the property accounting system remains the same regardless of the values used. Thus property accounting allows one to sidestep the valuation controversies of accounting and to describe the underlying transactions in an objective manner.

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