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# CATEGORY THEORY AND CONCRETE UNIVERSALS

## 1. INTRODUCTION

This paper presents the notion of a *theory of universals*. Set theory is shown to be a theory of universals. The Cantor-Frege naive set theory allowed sets as universals for a property to qualify as concrete instances of the property. This led to the set theoretical paradoxes. Axiomatic set theory was reconstructed using an iterative concept of a set so the set as the universal for a property was more abstract than the instances. Set theory, based on the iterative concept of sets, is thus the theory of *abstract* universals. A separate theory is needed for *concrete* universals.

This paper argues that category theory is the theory of concrete universals. The notion of the concrete universal allows the systematic philosophico-logical interpretation of the universal mapping properties of category theory.

Moreover, category theory provides a precise mathematical model for the self-predicative version of Plato's Theory of Forms. It rigorously models many of the ancient philosophical ideas about universals such as: (1) the Platonic notion that all the instances of a property have the property by virtue of participating in the universal and (2) the notion of the universal as showing the essence of a property without any imperfections.

## 2. THEORIES OF UNIVERSALS

In Plato's Theory of Ideas or Forms ($\epsilon\iota\delta\eta$), a property $F$ has an entity associated with it, the ***universal*** $u_F$, which uniquely represents the property. An object $x$ has the property $F$, i.e., $F(x)$, if and only if (iff) the object $x$ ***participates*** in the universal $u_F$. Let $\mu$ (from $\mu\epsilon\theta\epsilon\xi\iota\zeta$ or methexis) represent the participation relation so

$$\text{“}x\ \mu\ u_F\text{” reads as “}x \text{ participates in } u_F\text{”.}$$

Given a relation $\mu$, an entity $u_F$ is said to be ***a universal for the property***

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$F$ (with respect to $\mu$) if it satisfies the following *universality condition*:

$$\text{for any } x,\ x\,\mu\,u_F \text{ if and only if } F(x).$$

A universal representing a property should be in some sense unique. Hence there should be an equivalence relation ($\approx$) so that universals satisfy a *uniqueness condition*:

$$\text{if } u_F \text{ and } u_{F'} \text{ are universals for the same } F, \text{ then } u_F \approx u_{F'}.$$

A mathematical theory is said to be a *theory of universals* if it contains a binary relation $\mu$ and an equivalence relation $\approx$ so that with certain properties $F$ there are associated entities $u_F$ satisfying the following conditions:

(I) *Universality*: for any $x$, $x\mu u_F$ iff $F(x)$, and
(II) *Uniqueness*: if $u_F$ and $u_{F'}$ are universals for the same $F$ [i.e., satisfy (I) above], then $u_F \approx u_{F'}$.

A universal $u_F$ is said to be *abstract* if it does not participate in itself, i.e., $\neg(u_F\,\mu\,u_F)$. Alternatively, a universal $u_F$ is *concrete* if it is self-participating, i.e., $u_F\,\mu\,u_F$.

## 3. SET THEORY AS THE THEORY OF ABSTRACT UNIVERSALS

There is a modern mathematical theory which readily qualifies as a theory of universals, namely *set theory*. The universal representing a property $F$ is the *set* of all elements with the property:

$$u_F = \{x \mid F(x)\}.$$

The participation relation is the set membership relation usually represented by $\in$. The universality condition in set theory is the equivalence called a (naive) *comprehension axiom*: there is a set $y$ such that

$$\text{for any } x,\ x \in y \text{ iff } F(x).$$

Set theory also has an *extensionality axiom* which states that two sets with the same members are identical:

$$\text{for all } x,\ (x \in y \text{ iff } x \in y') \text{ implies } y = y'.$$

Thus if $y$ and $y'$ both satisfy the comprehension axiom scheme for the same $F$ then $y$ and $y'$ have the same members so $y = y'$. Hence in set theory the uniqueness condition on universals is satisfied with the equivalence relation ($\approx$) as equality ($=$) between sets. Thus naive set theory qualifies as a theory of universals.

The hope that naive set theory would provide a *general* theory of universals proved to be unfounded. The naive comprehension axiom lead to inconsistency for such properties as

$$F(x) \equiv x \text{ is not a member of } x \equiv x \notin x.$$

If $R$ is the universal for that property, i.e., $R$ is the set of all sets which are not members of themselves, the naive comprehension axiom yields the contradiction known as *Russell's Paradox*:

$$R \in R \text{ iff } R \notin R.$$

The characteristic feature of Russell's Paradox and the other set theoretical paradoxes is the self-reference wherein the universal is allowed to qualify for the property represented by the universal, e.g., the Russell set $R$ is allowed to be one of the $x$'s in the universality relation

$$x \in R \text{ iff } x \notin x.$$

There are several ways to restrict the naive comprehension axiom to defeat the set theoretical paradoxes, e.g., as in Russell's type theory, Zermelo-Fraenkel set theory, or von Neumann-Bernays set theory. The various restrictions are based on an iterative concept of set (Boolos, 1971) which forces a set $y$ to be more 'abstract', e.g., of higher type or rank, than the elements $x \in y$. Thus the universals provided by the various set theories are 'abstract' universals in the intuitive sense that they are more abstract than the objects having the property represented by the universal. Sets may not be members of themselves. Quine's system ML (1955b) allows "$V \in V$" for the universal class $V$, but no standard model of ML has ever been found where "$\in$" is interpreted as set membership (viz., Hatcher, 1982, chap. 7). We are concerned with theories which are "set theories" in the sense that "$\in$" can be interpreted as set membership.

With the modifications to avoid the paradoxes, a set theory still qualifies as a theory of universals. The membership relation is the participation relation so that for suitably restricted predicates, there exists a set satisfying the universality condition. Set equality serves as the equivalence relation in the uniqueness conditions. But set theory cannot qualify as a *general* theory of universals. The paradox-induced modifications turn the various set theories into theories of *abstract* (i.e., non-self-participating) universals since they prohibit the self-membership of sets.

## 4. CONCRETE UNIVERSALS

Philosophy contemplates another type of universal, a *concrete universal*. The intuitive idea of a concrete universal for a property is that it is an object which has the property and has it in such a universal sense that all other objects with the property resemble or participate in that paradigmatic or archetypical instance. The concrete universal $u_F$ for a property $F$ is *concrete* in the sense that it has the property itself, i.e., $F(u_F)$. It is *universal* in the intuitive sense that it represents $F$-ness in such a perfect and exemplary manner that any object resembles or participates in the universal $u_F$ if and only if it has the property $F$.

The intuitive notion of a concrete universal occurs in ordinary language (the 'all-American boy'), in theology ("the Word made flesh"), in the arts and literature (the old idea that great art uses a concrete instance to universally exemplify certain human conditions), and in philosophy (the perfect example of $F$-ness with no imperfections, only those attributes necessary for $F$-ness).

Did the notion of a concrete universal occur in Plato's Theory of Forms? Plato's forms are often considered to be abstract or non-self-participating universals quite distinct and 'above' the concrete instances. In the words of one Plato scholar, "not even God can scratch Doghood behind the Ears" (Allen, 1960). But Plato did give examples of self-participation or self-predication, e.g., that Justic is just [Protagoras 330]. Moreover, Plato often used expressions that indicated self-predication of universals.

> But Plato also used language which suggests not only that the Forms exist separately (χωριστα) from all the particulars, but also that each Form is a peculiarly accurate or good particular of its own kind, i.e., the standard particular of the kind in question or the model (παραδειγμα) to which other particulars approximate. (Kneale and Kneale, 1962, p. 19).

But many scholars regard the notion of a Form as *paradeigma* or concrete universal as an error.

For general characters are not characterized by themselves: humanity is not human. The mistake is encouraged by the fact that in Greek the same phrase may signify both the concrete and the abstract, e.g. *το λευκον* (literally 'the white') both 'the white thing' and 'whiteness', so that it is doubtful whether *αυτο το λευκον* (literally 'the white itself') means 'the superlatively white thing' or 'whiteness in abstraction'. (Kneale and Kneale, 1962, pp. 19–20]

Thus some Platonic language is ambivalent between interpreting a form as a concrete universal ('the superlatively white thing') and an abstract universal ('whiteness in abstraction').

The literature on Plato has reached no resolution on the question of self-predication. Scholarship has left Plato on both sides of the fence; many universals are not self-participating but some are. It is fitting that Plato should exhibit this ambivalence since the self-predication issue has only come to a head in this century with the set theoretical antinomies. Set theory had to be reconstructed as a theory of universals which were rigidly non-self-participating.

The reconstruction of set theory as the theory of abstract universals cleared the ground for a separate theory of universals that are always self-participating. Such a theory of concrete universals would realize the self-predicative strand of Plato's Theory of Forms.

A theory of concrete universals would have an appropriate participating relation $\mu$ so that for certain properties $F$, there are entities $u_F$ satisfying the universality condition:

$$\text{for any } x,\ x \,\mu\, u_F \text{ if and only if } F(x).$$

The universality condition and $F(u_F)$ imply that $u_F$ is a *concrete* universal in the previously-defined sense of being self-participating, $u_F \,\mu\, u_F$. A theory of concrete universals would also have to have an equivalence relation so the concrete universals for the same property would be unique up to that equivalence relation.

Is there a precise mathematical theory of concrete universals? Is there a theory that is to concrete universals as set theory is to abstract universals? Our claim is that *category theory* is precisely that theory. Before the more technical development, a simple example will illustrate the concrete universals in category theory.

A partially-ordered set is a simple example of a category. Conisder a universe of sets with the inclusion relation $\leq$ as the partial ordering relation. Given sets a and b, consider the property

$$F(x) \equiv x \leq a \,\&\, x \leq b.$$

The participation relation is set inclusion $\leq$ and the intersection $a \cap b$ is the universal $u_F$ for this property $F(x)$. The universality relation states that the intersection is the greatest lower bound of $a$ and $b$ in the inclusion ordering:

$$\text{for any } x,\ x \leq a \cap b \text{ iff } x \leq a \,\&\, x \leq b.$$

The universal has the property it represents, i.e., $a \cap b \leq a \,\&\, a \cap b \leq b$, so it is a self-participating or concrete universal. Two concrete universals for the same property must participate in each other. In partially ordered sets, the anti-symmetry condition

$$y \leq y' \,\&\, y' \leq y \text{ implies } y = y'$$

means that equality can serve as the equivalence relation in the uniqueness condition for universals in a partial order.

## 5. CATEGORY THEORY: SOME DEFINITIONS

Some familiarity is assumed with Eilenberg and MacLane's theory of categories (e.g., MacLane and Birkhoff, 1967 or MacLane, 1971) although some elementary definitions will be given to establish notation.

A *category* C consists of

(a) a set of *objects* $a, b, c, \ldots,$

(b) for each pair of objects $\langle a, b \rangle$, a set $\hom_C(a, b) = C(a, b)$ whose elements are represented as *arrows* or *morphsims* $f\colon a \to b$,

(c) for any $f \in \hom_C(a, b)$ and $g \in \hom_C(b, c)$, there is the *composition* gf: $a \to b \to c$ in $\hom_C(a, c)$,

(d) composition of arrows is an associative operation, and

(e) for each object $a$, there is an arrow $1_a \in \hom_C(a, a)$, called the *identity* of $a$, such that for any $f\colon a \to b$ and $g\colon c \to a$, $f1_a = f$ and $1_a g = g$.

An arrow $f\colon a \to b$ is an *isomorphism* if there is an arrow $g\colon b \to a$ such that $fg = 1_b$ and $gf = 1_a$.

## 6. CATEGORY THEORY AS THE THEORY OF CONCRETE UNIVERSALS

For the concrete universals of category theory, the *participation relation* is the *uniquely-factors-through* relation. It can always be formulated in a suitable category as:

$$\text{“}x\,\mu\,u\text{” means “there exists a unique arrow } x \to u\text{”.}$$

Then $x$ is said to *uniquely factor through u*, and the arrow $x \to u$ is the unique factor or participation morphism. In the universality condition,

$$\text{for any } x,\ x\,\mu\,u \text{ if and only if } F(x),$$

the existence of the identity arrow $1_u\colon u \to u$ is the self-participation of the concrete universal which corresponds with $F(u)$, the application of the property to $u$. It is sometimes convenient to "turn the arrows around" and use the dual definition where "$x\,\mu\,u$" means "there exists a unique arrow $u \to x$" which can also be viewed as the original definition stated in the dual or opposite category.

In category theory, the *equivalence relation* used in the uniqueness condition is the *isomorphism* ($\cong$). Thus it must be verified that two concrete universals for the same property are isomorphic. By the universality condition, two concrete universals $u$ and $u'$ for the same property must participate in each other. Let $f : u' \to u$ and $g\colon u \to u'$ be the unique arrows given by the mutual participation. Then by composition $gf\colon u' \to u'$ is the unique arrow $u' \to u'$ but $1_{u'}$ is another such arrow so by uniqueness, $gf = 1_{u'}$. Similarly, $fg\colon u \to u$ is the unique self-participation arrow for $u$ so $fg = 1_u$. Thus mutual participation of $u$ and $u'$ implies $u \cong u'$.

Category theory therefore qualifies as a theory of universals with participation defined as "uniquely factors through" and the equivalence relation taken as isomorphism. The universal of category theory are self-participating or concrete; a universal $u$ uniquely factors through itself by the identity morphism $1_u$.

Category theory as the theory of concrete universals has quite a different flavor from set theory, the theory of abstract universals. Given the collection of all the elements with a property, set theory can *postulate* a more abstract entity, the set of those elements, to be universal. But category theory cannot postulate its universals because those universals are concrete. Category theory must *find* its universals, if at all, among the entities with the property.

## 7. AN EXAMPLE: PRODUCTS AS UNIVERSALS

The previous example of set intersection is a special case of the notion of a product in a category. In the general case, it becomes clear that the entities of interest are the morphisms more than the objects which appear as the domains and codomains of morphisms. Given objects $a$ and $b$ in a category $C$ (which the reader could take as the category $S$ of sets), the property $F$ applies to pairs of morphisms with a common domain and with $a$ and $b$ as the codomains:

$$F(\langle f, g\rangle) \text{ means for some } x,\ f\colon x \to a \text{ and } g\colon x \to b.$$

The universal for this property, if it exists, is a pair of morphisms $\pi_1\colon a \times b \to a$ and $\pi_2\colon a \times b \to b$ with a common domain, usually denoted $a \times b$, such that any other pair of morphisms $f\colon x \to a$ and $g\colon x \to b$ factor uniquely through the projections $\langle \pi_1, \pi_2\rangle$. "Factor uniquely through" means there exists a unique morphism $h\colon x \to a \times b$ such that the triangles commute in the following diagram (Figure 1):

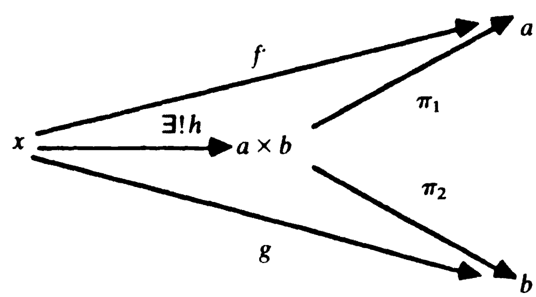


Fig. 1.

As a universality condition, this is:

> $\langle f, g\rangle$ uniquely factors through $\langle \pi_1, \pi_2\rangle$ iff for some $x$, $f\colon x \to a$ and $g\colon x \to b$.

If $\langle f, g\rangle$ has the property, $\langle f, g\rangle$ uniquely factors through $\langle \pi_1, \pi_2\rangle$. And if $\langle f, g\rangle$ uniquely factors through (i.e., participates in) $\langle \pi_1, \pi_2\rangle$, the property is *reflected* back from the concrete universal $\langle \pi_1, \pi_2\rangle$ to the entities participating in it. That is, if $f = \pi_1 h$ and $g = \pi_2 h$ for some factor map $h: x \to a \times b$, then $f$ and $g$ must have the property of being morphisms from some common domain to $a$ and $b$ (since $\pi_1$ and $\pi_2$ have that property). Thus the Platonic notion that an entity has *F by virtue of* participating in the universal for $F$ has a precise realization in category theory.

The previous definition of a universal for a property $F$ as an *object u* was the simplest definition to yield the result that two universals for the same property are isomorphic. In most cases of interest, the universals will be morphisms or collections of morphisms. It is always possible to define new categories so these collections of morphisms will become the objects in the new category. In the product example, consider a new category $C'$ where the objects are pairs $\langle f_1, f_2\rangle$ of morphisms from $C$ with a common domain. Given two objects $\langle f_1, f_2\rangle$ and $\langle g_1, g_2\rangle$ with codomain $(f_i)$ = codomain $(g_i)$ for $i = 1, 2$, a morphism

$$h: \langle f_1, f_2\rangle \to \langle g_1, g_2\rangle$$

in $C'$ is a morphism in $C$ from the domain of the $f$'s to the domain of the $g$'s such that $g_1 h = f_1$ and $g_2 h = f_2$. In this new category $C'$, the pair $\langle \pi_1, \pi_2\rangle$ fits the definition of a universal object:

$$\text{for any } \langle f_1, f_2\rangle \text{ in } C', \exists!\ \langle f_1, f_2\rangle \to \langle \pi_1, \pi_2\rangle \text{ iff } F(\langle f_1, f_2\rangle).$$

## 8. AN IMPORTANT EXAMPLE: THE NATURAL NUMBERS AS A UNIVERSAL

As the theory of concrete universals, category theory does not try to reconstruct arithmetic or analysis or other branches of mathematics from the ground up as does set theory. Abstract universals can be "made" (e.g., the construction of the Zermelo hierarchy) whereas concrete universals must be "found" (as the universal paradigm example in the already-existing class of entities having a property). The foundational role of category theory (qua theory of concrete universals) is to *characterize what is important in mathematics by exhibiting its concrete universality properties*, not to provide some alternative construction of the same entities.

For instance, why are the Natural Numbers important? From the viewpoint of set-theoretical model theory, the Natural Numbers are one among infinites of other models of a language containing a designated element (zero in N) and a unary function (the successor function in N). Instead of providing yet another construction of the Natural Numbers, category theory gives the universality property which shows why they are important. Lawvere (e.g., MacLane and Birkhoff, 1968, p. 67) has shown that the (second-order) Peano axioms are equivalent to a certain universal mapping property which exhibits the *Natural Numbers as the universal enumerating-set*.

Peano's Axioms: $N$ is a set with a designated element 0 and a function $\sigma: N \to N$ such that

(1) $\sigma$ is injective,
(2) 0 is not in the image $\sigma(N)$, and
(3) for any subset $P$ of $N$, if
(a) 0 is in $P$, and
(b) for all $n$ in $N$, if $n \in P$ implies $\sigma(n) \in P$
then $P = N$ (induction principle).

Let $C$ be the category whose objects are sets $S$ endowed with a designated element $z$ and an endofunction $f: S \to S$ and whose morphisms are set functions that preserve the designated elements and commute with the endofunctions. Thus if $\langle S, z, f \rangle$ and $\langle S', z', f' \rangle$ are in $C$, then a morphism $\langle S, z, f \rangle \to \langle S', z', f' \rangle$ is given by a function $g: S \to S'$ such that $g(z) = z'$ and for any $s$ in $S$, $f'(g(s)) = g(f(s))$. The universality property which characterizes the Natural Numbers is the:

> *Peano-Lawvere Axiom*: $\langle N, 0, \sigma \rangle$ is an initial object in the category $C$, i.e., for any object $\langle S, z, f \rangle$ in $C$, there is a unique morphism $g: \langle N, 0, \sigma \rangle \to \langle S, z, f \rangle$.

The participation relation is that:

> $\langle S', z', f' \rangle \, \mu \langle S, z, f \rangle$ iff there is a unique morphism $\langle S, z, f \rangle \to \langle S', z', f' \rangle$.

The property $F(\langle S, z, f \rangle)$ is that:

> there is a sequence $s_0, s_1, \ldots$ in $S$ such that $s_0 = z$ and $f$ gives the next element in the sequence, i.e., $f(s_n) = s_{n+1}$ for all $n$ (note that the Natural Numbers are used implicitly or explicitly in the language stating the property).

The Natural Numbers $\langle N, 0, \sigma \rangle$ are the concrete universal for that property $F$, they are the universal enumerating set. Given any object $\langle S, z, f \rangle$ in $C$, there exists a unique morphism $g: \langle N, 0, \sigma \rangle \rightarrow \langle S, z, f \rangle$. The unique participation map is given by the iterates of $f$ applied to $z$, i.e., $g(n) = f^n(z)$ where $g(0) = z$. The sequence $s_0, s_1, \ldots$ in $S$ is given by the image of $g$.

Peano's Axioms can be derived from the universal property of the Natural Numbers (e.g., MacLane and Birkhoff, 1968, chap. II, §11). Consider, for example, the induction principle. Let $P$ be a subset of $N$, such that (a) 0 is in $P$, and (b) for all $n$ in $N$, if $n \in P$ then $\sigma(n) \in P$. Then $f(n) = \sigma(n)$ defines a function $f: P \rightarrow P$ so $\langle P, 0, f \rangle$ is an object in $C$. Hence there is a unique participation map $s: N \rightarrow P$ such that the triangle and rectangle in the following diagram (Figure 2) commutes.

$$
\begin{tikzcd}
 & N \arrow[r, "\sigma"] \arrow[d, "s"] & N \arrow[d, "s"] \\
1 \arrow[ur, "0"] \arrow[r, "0"'] & P \arrow[r, "f"] & P
\end{tikzcd}
$$

Fig. 2.

The inclusion map $j: P \rightarrow N$ makes the bottom triangle and rectangle in the following diagram (Figure 3) commute.

$$
\begin{tikzcd}
 & N \arrow[r, "\sigma"] \arrow[d, "s"] & N \arrow[d, "s"] \\
1 \arrow[ur, "0"] \arrow[r, "0"] \arrow[dr, "0"'] & P \arrow[r, "f"] \arrow[d, "i"] & P \arrow[d, "i"] \\
 & N \arrow[r, "\sigma"] & N
\end{tikzcd}
$$

Fig. 3.

Hence the outer triangle and rectangle commute so $js: N \rightarrow N$ is the unique self-participation map for $N$, i.e., $js$ is the identity map $N \rightarrow N$. Therefore the inclusion map $j: P \rightarrow N$ must be surjective, i.e., $P = N$. Thus the induction principle-like the other Peano Axioms-can be derived from the universal property of the Natural Numbers.

## 9. UNIVERSALS AS ESSENCES

The concrete universal for a property represents the essential characteristics of the property without any imperfections (to use some language of an Aristotelian stamp). *All* the objects in category theory with universal mapping properties such as limits and colimits (vis. Schubert, 1972, chaps. 7–8) are concrete universals for universal properties. Thus the universals of category theory can typically be presented as the limit (or 'colimit') of a process of filtering out or eliminating imperfections to arrive at the pure essence of the property.

Consider the previous example of the intersection $a \cap b$ of sets $a$ and $b$ as the concrete universal for the property $F$ of being contained in $a$ and in $b$. If a set $x$ has the property but is not the universal, then $x$ has certain 'imperfections' relative to the property $F$. In this case, the imperfections of $x$ are the sets $x'$ which are contained in $a$ and in $b$, but which are not contained in $x$. If we remove all the imperfections, i.e., add to $x$ all the other elements common to $a$ and $b$, then we arrive at the 'essence' of the property, the concrete universal $a \cap b$ for the property.

This limiting process of arriving at the universal can be expressed in a more categorical fashion. In category theory, it is useful to consider the *factors-through* relation. Since participation is the *uniquely*-factors-through relation, factors-through can be thought of as a "weak-participation" relation. Given the uniquely-factors-through relation $\mu$ for a property $F$, the factors-through relation will *reflect* the property in the sense that:

$$\text{if } F(y) \text{ and } x \text{ factors through } y, \text{ then } F(x).$$

If $x$ and $y$ are instances of $F$ and $x$ factors through $y$, then $y$ is said to be *equally or more essential* than $x$ (with respect to $F$). In other words, the weak-participation relation for $F$ can be considered as the 'essentialness' preordering (reflexive and transitive relation) on the instances of $F$. Then the concrete universal for $F$ would, by definition, be equally or more essential than all the instances of $F$, i.e., 'the essence' of $F$.

In a preordering as a category, factors-through is the same as participation since there is at most one morphism between two objects. In the intersection example, the inclusion relation $\leq$ is the participation relation. The property $F(x) \equiv x \leq a \;\&\; x \leq b$ is preserved under arbitrary unions, i.e.,

$$\text{if } F(x_\beta) \text{ for any } x_\beta \text{ in } \{x_\beta \mid \beta \in B\}, \text{ then } F(\bigcap_\beta x_\beta).$$

Hence given any collection of instances $\{x_\beta \mid \beta \in B\}$ of the property $F$, their union is equally or more essentially $F$ than the instances. None of the sets in the collection are imperfections of the union. Thus the limit of this process, the 'essence of $F$-ness', can be obtained as the union of *all* the instances of $F$:

$$\bigcup\{x \mid x \leq \;\&\; x \leq b\} = a \cap b.$$

It has *no* imperfections relative to the property $F$. Moreover, since the universal is concrete, the set $a \cap b$ is among the sets $x$ involved in the union and it contains all the other such sets $x$. Thus the union is 'taken on', i.e., is equal to one of the sets in the union.

All the category theory examples can be dualized by 'reversing the arrows'. Reversing the inclusion relation in the definition of $F$ yields the property:

$$G(x) \equiv a \leq x \;\&\; b \leq x.$$

The participation relation $\mu$ for $G$ is the reverse of inclusion $\geq$ and the union of $a$ and $b$ is the concrete universal. The universality condition is:

$$\text{for all } x,\; x \geq a \cap b \text{ iff } a \leq x \;\&\; b \leq x.$$

If $x$ has the property $G$ but is not the univ,ersal, then $x$ has certain 'imperfections'. An imperfection of $x$ (relative to the $G$ property) would be given by an another set $x'$ containing both $a$ and $b$ but not containing $x$. A set of instances of $G$ could be purified of some imperfections by taking the intersection of the set. $G$-ness is preserved under arbitrary intersections. The intersection of a collection of sets with the property $G$ is equally or more essential than the sets in the collection. None of the sets in the collection are imperfections of the intersection. Thus the universal or essence of $G$-ness can be obtained as the intersection of *all* the sets with the property $G$;

$$\bigcap\{x \mid a \leq x \ \& \ b \leq x\} = a \cap b.$$

It has no imperfections relative to $G$.

## 10. LIMITS AND COLIMITS AS ESSENCES

The intersection and union of sets in an inclusion ordering are examples of limits and colimits in categories. All limits and colimits are concrete universals for certain defining properties. The result that the intersection or limit $a \cap b$ can be obtained as the union or colimit of all the instances of the defining property extends to all limits and colimits. Any limit (respectively, colimit) is the colimit (limit) of the instances of its defining property. In philosophical terms, the limit or colimit is the essence arrived at by the limiting process of purifying all the instances of the property of their imperfections.

This result will be illustrated using a limit that is a special case of the pullback construction. Instead of considering just pairs of morphisms into $a$ and $b$ as in the product example, suppose there is an additional morphism $j: a \to b$. Let the property $G$ apply to a pair of morphisms $\langle f, g \rangle$ where $f: x \to a$ and $g: x \to b$ if $\langle f, g \rangle$ *commutes with* $j$ in the sense that $jf: x \to a \to b = g: x \to b$. Given a pair $\langle f', g' \rangle$ of morphisms into $a$ and $b$ with the common domain $x'$, a morphism $h: \langle f, g \rangle \to \langle f', g' \rangle$ is defined by a morphism $h: x \to x'$ such that $f = f'h$ and $g = g'h$. Then $\langle f, g \rangle$ is said to *factor through* $\langle f', g' \rangle$. If the factorization is unique, it is the participation relation $\mu$ for the property $G$.

The property $G$ of commuting with $j$ is *reflected* by these morphisms $h$. If $\langle f', g' \rangle$ commutes with $j$ and there is a morphism $h: \langle f, g \rangle \to \langle f', g' \rangle$, then $\langle f, g \rangle$ commutes with $j$ since $jf' = g'$ implies $jf'h = g'h$ or $jf = g$. In that case, $\langle f', g' \rangle$ is said to be *equally or more essential* than $\langle f, g \rangle$ with respect to the property $G$. Thus the factors-through relation defines the essentialness preorder on the instances of $G$.

Given an instance of $G$, $\langle f, g \rangle$, an *imperfection* of $\langle f, g \rangle$ (relative to $G$) is another $G$-instance $\langle f', g' \rangle$ which does *not* factor through $\langle f, g \rangle$.

The concrete universal for $G$, if it exists in $C$, is given by a pair of morphisms $\langle p_a, p_b \rangle$ into $a$ and $b$ with a common domain denoted lim $G$. That pair commutes with $j$ and is universal among pairs of morphisms into $a$ and $b$ with a common domain which commute with $j$. Thus given another pair $\langle f, g \rangle$ with that property $G$, there is a unique factor morphism $i: x \to \lim G$ such that $p_a i = f$ and $p_b i = g$ as indicated in the following commutative diagram (Figure 4).

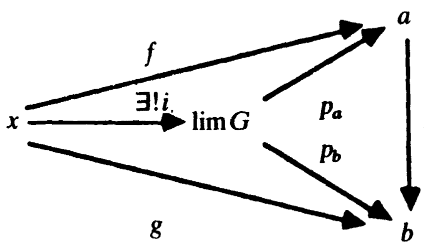


Fig. 4.

The concrete universal for $G$ has no imperfections relative to $G$.

The aim is to show that a colimit of instances of the property $G$ also has $G$ and is equally or more essential than the instances. Then the limit $\lim G$ can be obtained as the colimit of all the instances of $G$. Consider any set of instances $\langle f, g\rangle$ of $G$ with morphisms $h: \langle f, g\rangle \to \langle f', g'\rangle$ as previously defined. The common domains of the pairs such as $x$ and $x'$ together with the morphisms $h: x \to x'$ define a *diagram D* in the category $C$ [e.g., Arbib and Manes, 1975, p. 45; or Schubert, 1972, chap. 6]. The *colimit* of the diagram is given by an object $\operatorname{colim} D$ and a set of morphisms $i_x: x \to \operatorname{colim} D$ for $x$ in the diagram. The morphisms $i_x$ commute with the diagram morphisms $h$ in the sense that $i_{x'}h = i_x$, and the colimit is the universal such set of morphisms. The $f$ morphisms from the pairs $\langle f, g\rangle$ form another such set of morphisms commuting with the $h$ morphisms in the diagram $D$ so there exists a unique factor morphism $i_a$: colim $D \to a$ such that for all the $f$'s, $i_a i_x = f$, i.e., all the triangles in the following diagram (Figure 5) commute.

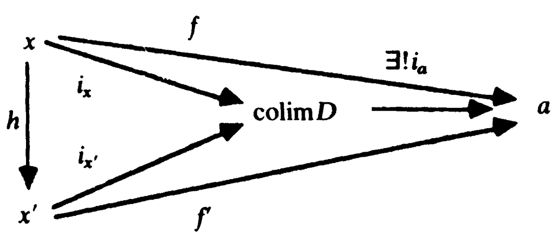


Fig. 5.

The $g$ morphisms from the pairs $\langle f, g\rangle$ for another such set of morphisms so there exists a unique morphism $i_b$: $\operatorname{colim} D \to b$ such that for all the $g$'s, $i_b i_x = g$.

It remains to show that the pair $\langle i_a, i_b\rangle$ has the property $G$ of commuting with $j$. For any $x$ in the diagram $D$, $g = jf = j(i_a i_x) = (j i_a) i_x$.

But $i_b$ is the unique morphism such that for any $x$ in the diagram, $i_b i_x = g$ so $i_b = j i_a$, i.e., $\langle i_a, i_b \rangle$ commutes with $j$. The morphism $i_x: x \rightarrow \operatorname{colim} D$ shows that $\langle f, g \rangle$ factors through $\langle i_a, i_b \rangle$ for any $x$ in the diagram, so the colimit of any set of instances of $G$ yields an instance of $G$ that the equally or more essential than those instances. None of those instances are imperfections of $\operatorname{colim} D$ relative to $G$.

The limit $\lim G$ can be obtained as the colimit of the diagram formed by *all* the instances of $G$. It would have no imperfections relative to $G$. The collection of all the instances of $G$ does not necessarily form a small set, but the colimit exists, if $\lim G$ exists, since $\lim G$ would be a terminal object in the diagram. The colimit is 'taken on' by $\lim G$. This example illustrates the general theme that the limits and colimits of category theory can be obtained as the result of 'purifying' all the instances of the defining property $G$ of their imperfections to arrive at the 'essence' of $G$-ness.

## 11. THE THIRD MAN ARGUMENT IN PLATO

Much of the modern Platonic literature on self-participation stems from Vlastos' work on the Third Man argument (1954, 1981). The name derives from Aristotle, but the argument occurs in the dialogues.

> But now take largeness itself and the other things which are large. Suppose you look at all these in the same way in your mind's eye, will not yet another unity make its appearance—a largeness by virtue of which they all appear large?
>
> So, it would seem.
>
> If so, a second form of largeness will present itself, over and above largeness itself and the things that share in it, and again, covering all these, yet another, which will make all of them large. So each of your forms will no longer be one, but an indefinite number. (Parmenides, p. 132).

If a form is self-predicative, the participation relation can be interpreted as 'resemblence'. An instance has the property $F$ because it resembles the paradigmatic example of $F$-ness. But then, the Third Man argument contends, the common property shared by Largeness and other large things give rise to a 'One over the many', a form Largeness* such that Largeness and the large things share the common property by virtue of resembling Largeness*. And the argument repeats itself giving rise to infinite regress of forms. A key part of the Third Man argument is what Vlastos calls the *Non-Identity thesis*:

NI If anything has a given character by participating in a Form, it is not identical with that Form. (Vlastos, 1981, p. 351)

It implies that Largeness* is not identical with Largeness.

P. T. Geach (1956) has developed a self-predicative interpretation of Forms as standards or norms, an idea he attributes to Wittgenstein. A stick is a yard long because it resembles, length-wise, the Standard Yard measure. Geach avoids the Third Man regress with the exceptionalist device of holding the Form 'separate' from the many so they could not be grouped together to give rise to a new 'One over the many'. Geach aptly notes the analogy with Frege's ad hoc and unsuccessful attempt to avoid the Russell-type paradoxes by allowing a set of all and only the sets which are not members of themselves–except for that set itself (viz., Quine, 1955a; Geach, 1980).

Category theory provides a mathematical model for the Third Man argument, and it shows how to avoid the regress. In the last section, it was shown that given a collection or diagram $D$ of entities with a certain property $F$, an entity colim$D$ could be constructed that was equally or more essentially $F$ than the entities in the collection. That is the mathematical model for the process of forming the 'One over the many'. The One (colim$D$) has the property $F$ shared by the many, and the many participate in the One. A new collection or diagram $D^*$ could be formed using the many and the One, and thus a new One* (colim$D^*$) could be formed. In this manner, category theory rigorously models the Third Man argument.

The category theoretic model shows that the flaw in the Third Man argument lies not in self-predication but in the Non-Identity thesis (viz., Vlastos, 1954, pp. 326–29). 'The One' is not necessarily 'over the many'; it can be (isomorphic to) one among the many. In mathematical terms, a colimit or limit can 'take on' one of the elements in the diagram. In the special case of sets ordered by inclusion, the union or intersection of a collection of sets is not necessarily distinct from the sets in the collection; it could be one among the many.

For example, let $A = \bigcup\{A_\beta\}$ be the One formed as the union of a collection of many sets $\{A_\beta\}$. Then add $A$ to the collection and form the new One* as

$$A^* = \bigcup\{A_\beta\} \cup A.$$

This operation leads to no Third Man regress since $A^* = A$. In general, if the colimit colim$D$ of a diagram $D$ is appended onto the diagram to form a new diagram $D^*$, then it becomes a 'terminal object' in the new diagram $D^*$ so colim $D^* \cong \text{colim} D$.

## 12. ENTAILMENT AS A RELATION BETWEEN UNIVERSALS

In Plato's Theory of Forms, a logical inference is valid because it follows the necessary connections between universals. Threeness entails oddness because the universal for threeness "brings-on" (επιφερει or epipherei, viz., Vlastos, 1981, p. 102; or Sayre, 1969, Part IV) or "shares-in" the universal for oddness. In a mathematical theory of universals, the "entailment" relation between universals is defined as follows: given universals $u_G$ and $u_F$,

$u_G$ *entails* $u_F$ if for any $x$, if $x \, \mu \, u_G$ then $x \, \mu \, u_F$.

In set theory, the participation relation $\mu$ is the membership relation $\in$ so the entailment relation between sets as abstract universals is the *inclusion* relation. Thus in set theory as the theory of abstract universals, the entailment relation (inclusion) between universals is *not* the same as the participation relation (membership). Considerable effort was expended in the history of logic to clearly understand the difference between inclusion and membership, e.g., between the copulas in "All roses are beautiful" and "The rose is beautiful."

In category theory, the participation relation $\mu$ is the uniquely-factors-through relation and the universals are self-participating. If $u_G$ entails $u_F$, then $x \, \mu \, u_G$ implies $x \, \mu \, u_F$. Since $u_G \, \mu \, u_G$, it follows that $u_G \, \mu \, u_F$. Conversely, if $u_G \, \mu \, u_F$, then $x \, \mu \, u_G$ implies that $x$ factors through $u_F$. But since factors-through reflects the property $F$, $x$ is an instance of $F$ and thus $x \, \mu \, u_F$. Hence if $u_G$ entails $u_F$, then $u_G \, \mu \, u_F$, and vice versa. Thus for the concrete universals of category theory,

Entailment relation = Participation relation restricted to universals.

To speak in a philosophical mode for illustrative purposes, let 'The Rose' and 'The Beautiful' be the concrete universals for the respective properties. In the theory of concrete universals, the universal statement "All roses are beautiful" and the singular statement "The Rose is beautiful" are *equivalent*. Both express the proposition that The Rose participates in (i.e., entails) The Beautiful, and that proposition is distinct from the statement "The rose is beautiful" (about a plant in my backyard).

For a category theoretic example of entailment (or participation) between universals, let $G(\langle f, g\rangle)$ be the property of being morphisms from some common domain into $a$ and $b$ that commute with a morphism $j: a \rightarrow b$, i.e., $jf = g$. Let $F(\langle f, g\rangle)$ be the property of simply being morphisms from a common domain into $a$ and $b$. Clearly the following logical implication holds:

$$\text{for any } \langle f, g\rangle, \text{ if } G(\langle f, g\rangle) \text{ then } F(\langle f, g\rangle).$$

By the universality conditions for the universals $u_G = \langle p_a, p_b\rangle$ and $u_F = \langle \pi_1, \pi_2\rangle$,

$$\textit{for any } \langle f, g\rangle, \text{ if } \langle f, g\rangle\ \mu\ \langle p_a, p_b\rangle, \text{ then } \langle f, g\rangle\ \mu\ \langle \pi_1, \pi_2\rangle,$$

so $\langle p_a, p_b\rangle$ entails $\langle \pi_1, \pi_2\rangle$. Since $\langle p_a, p_b\rangle$ participates in $\langle \pi_1, \pi_2\rangle$, there is a unique factor morphism $I; \langle p_a, p_b\rangle \rightarrow \langle \pi_1, \pi_2\rangle$. If the ambient category $C$ is the category of sets, the common domain $\lim G$ of $\langle p_a, p_b\rangle$ can be represented as the subset of the cartesian product $a \times b$ consisting of the ordered pairs $\langle x, j(x)\rangle$ for $x$ in the set $a$, so $i$ is the embedding $i: \lim G \rightarrow a \times b$.

## 13. CONCLUDING REMARKS

Whitehead described European philosophy as a series of footnotes to Plato, and the Theory of Forms was central to Plato's thought. The interpretation of category theory as the theory of concrete universals provides a rigorous self-predicative mathematical model for Plato's Theory of Forms.

Logic becomes concrete in category theory as the theory of concrete universals. A property $F$ can be realized concretely as an object which is the universal $u_F$. The fact that $x$ is an $F$-instance can be realized concretely by an object which the unique participation morphism $x \rightarrow u_F$. A universal implication $(x)(G(x) \Rightarrow F(x))$ can be realized concretely by an object which is the unique participation morphism $u_G \rightarrow u_F$ wherein one universal 'brings-on' or entails another universal.

What is the relevance of category theory to the foundations of mathematics? Today, this question might be answered by pointing to Lawvere and Tierney's *theory of topoi* (e.g., Lawvere, 1972; Lawvere et al., 1975; or Hatcher, 1982). Topos theory can be viewed as a categorically-formulated generalization of set theory to abstract sheaf theory. A set can be viewed as a sheaf of sets on the one-point space, and much of the machinery of set theory can be generalized to sheaves (e.g., the author's 1971 dissertation (1974) generalizing the ultraproduct construction to sheaves on a topological space). Since much of mathematics can be formulated in set theory, it can be reconstructed with many variations in topoi.

The concept of category theory as the logic of concrete universals presents quite a different picture of the foundational relevance of category theory. Topos theory is important in its own right as a generalization of set theory, but it does not exclusively capture category theory's foundational contribution. Concrete universals do not "generalize" abstract universals, so as the theory of concrete universals, category theory does not try to generalize set theory, the theory of abstract universals. Category theory presents the theory of the other type of universals, the self-participating or concrete universals.

Category theory is relevant to foundations in a different way than set theory. As the theory of concrete universals, category theory does not attempt to derive all of mathematics from a single theory. Instead, category theory's foundational relevance is that it provides universality concepts to characterize the important structures or forms throughout mathematics.

The Working Mathematician knows that the importance of category theory is that it provides a criterion of importance in mathematics. Category theory provides the concepts to isolate the universal instance from among all the instances of a property. The Concrete Universal is the most important instance of a property because it represents the property in a paradigmatic way. It shows the essence of the property without any imperfections. All other instances have the property by virtue of participating in the Concrete Universal.

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