Category Theory and Selectionist/Universal Mechanisms:

Adjoints and Brain Functors

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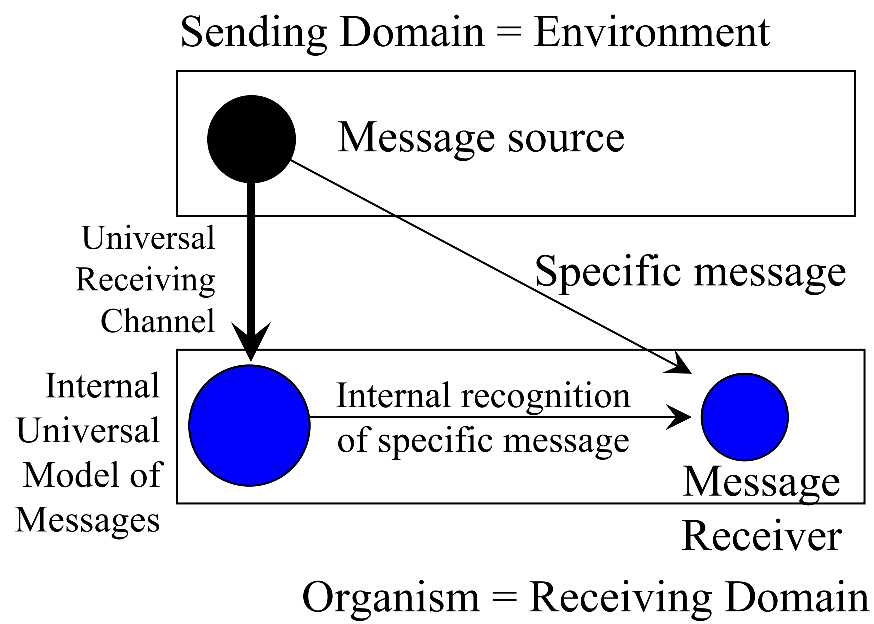
Motivation: Setting of the 'Selectionist' Problem

- Two domains, e.g., environment & organism.
- Message to be sent from sending to receiving domain.
- <u>Two</u> ways message might be received or recognized:

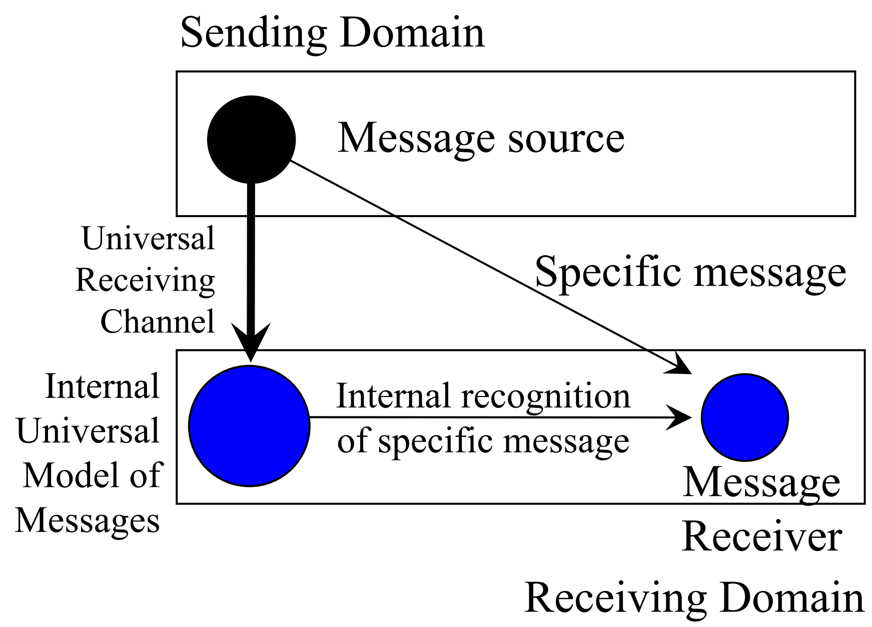


* <u>Instructionist</u>: direct transmission of detailed message;

* <u>Selectionist</u>: receiver has capacity to generate all 'possible' detailed messages and the 'sent message' is selected with relatively impoverished signal.

Edelman's 3 Components of Selectionist Mechanisms

- Generation of all 'possible' (or at least a diverse variety of) messages: "generator of diversity";
- Input from sender to select message from model of possible ones (polling);
- Recognition of selected message as the sent message (differential amplication).



Simplified "Toy" Examples

- Suit-making: Customer needs to send message of specifications of a suit that fits to the suit-maker:
  * Instructionist mechanism: customer makes available detailed measurements to tailor of a suit that fits;
  * Selectionist mechanism: suit-maker categorizes all 'possible' suit sizes so customer selects "42 long…".
- Sending a telegram: Customer needs to tell telegraph operator which message to send:
  * Instructionist: customer instructs operator with detailed message "Congratulations on the birth of your daughter";
  * Selectionist: operator constructs list of all 'possible' messages and customer selects "Message #6" which is "Congratulations on the birth of your daughter."

Examples of selectionist/universalist theories

Application	*Sender*	*Receiver*	*Universal internal model*	*Selection*
Selectionist evolution	Environment	Adapting organism	Population of variants generated in time	Selection of fit variants which reproduce more
Immune system	Antigens	Adapting antibody	Population of variant antibodies (low conc.)	Antigen selects its antibodies which multiply
Perception	External sensory signal	Perceiving organism	Variety of brain pre-image circuits at low amplitude	Signal resonates with a circuit which amplifies
Child language acquisition	Linguistic environment	Learning child	Universal language faculty	Experience determines how language faculty 'unfolds'

Chomskyan themes in universalist mechanisms: Message example

- Rich profligate internal structure: capacity to generate details of ‘all’ the possible messages;
- Impoverished or minimal inputs: selecting message #6 inputs less information that whole instructive message “Congratulations on the birth of your daughter”;
- Active role of internal mechanism: generates structure of all possible messages rather than passively receiving detailed message like a stamp in wax;
- Relative autonomy of internal mechanism: results from active internal role + minimal external input.

Category Theory & Adjoint Functors

- Category theory was first explicitly developed by Saunders MacLane and Samuel Eilenberg in 1945;
- Basic ideas were: categories, functors between categories, and natural transformations between functors.
- Adjoint functors were defined by Daniel Kan in 1958.
- Adjoints are now increasingly seen as fundamental not only to category theory but to foundations of mathematics.
- Adjoints characterize what is important and universal throughout mathematics itself.
- Are there empirical applications for such an important concept?

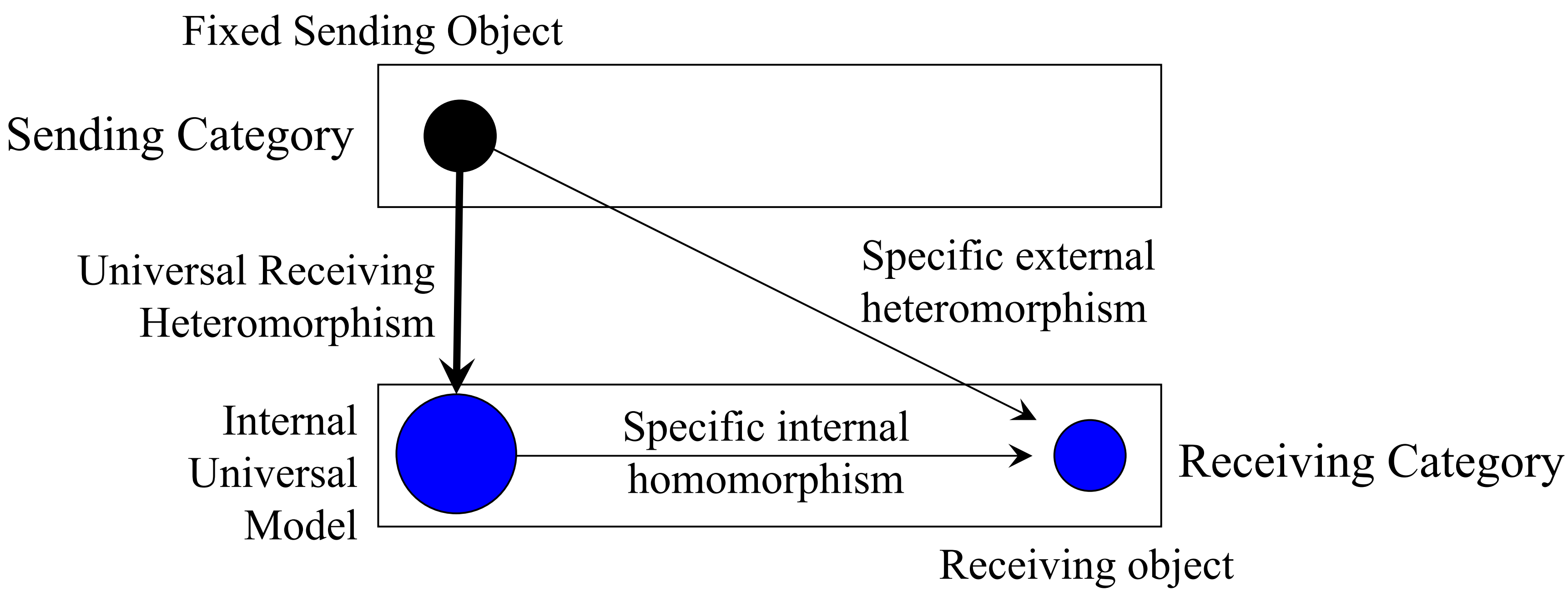
Adjoints in CT and Foundations

- "The notion of adjoint functor applies everything that we've learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics." (Steve Awodey, *Category Theory*)
- "The isolation and explication of the notion of adjointness is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas." (Robert Goldblatt, *Topoi*)
- "Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics." (Paul Taylor, *Practical Foundations of Mathematics*)

Adjoints suggesting models for universalist mechanisms

- If adjoints (universal mapping properties) are fundamental to characterize what is important in mathematics itself , then we might expect adjoints to give idealized models of important mechanisms in the empirical sciences, e.g., selectionist/universal mechanisms.
- A new treatment of adjoints–the "heteromorphic theory of adjoints"–allows exactly this application: adjoints (or rather two half-adjunctions) as abstract mathematical models for selectionist/universal mechanisms.

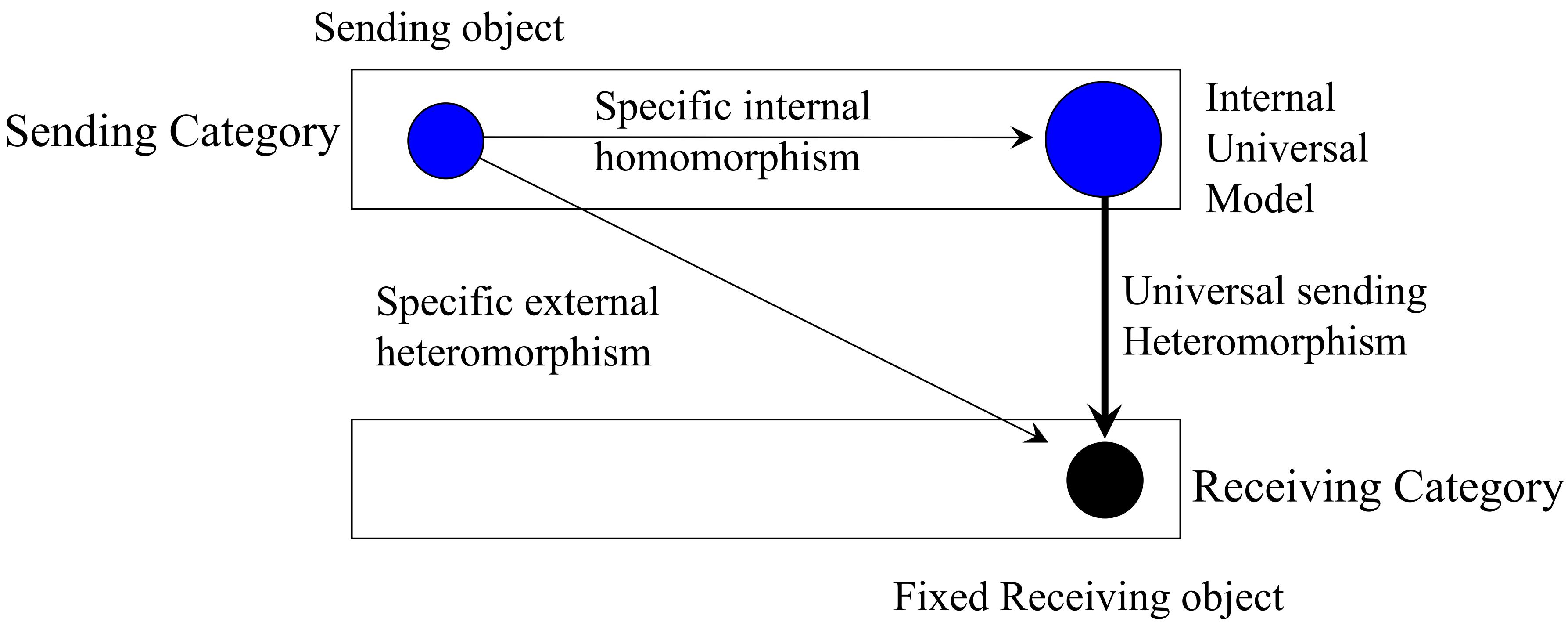
Left half-adjunction: “Recognition”



Left Half-adjunction (universal in receiving category)

- Note similarity to previous selectionist diagram.

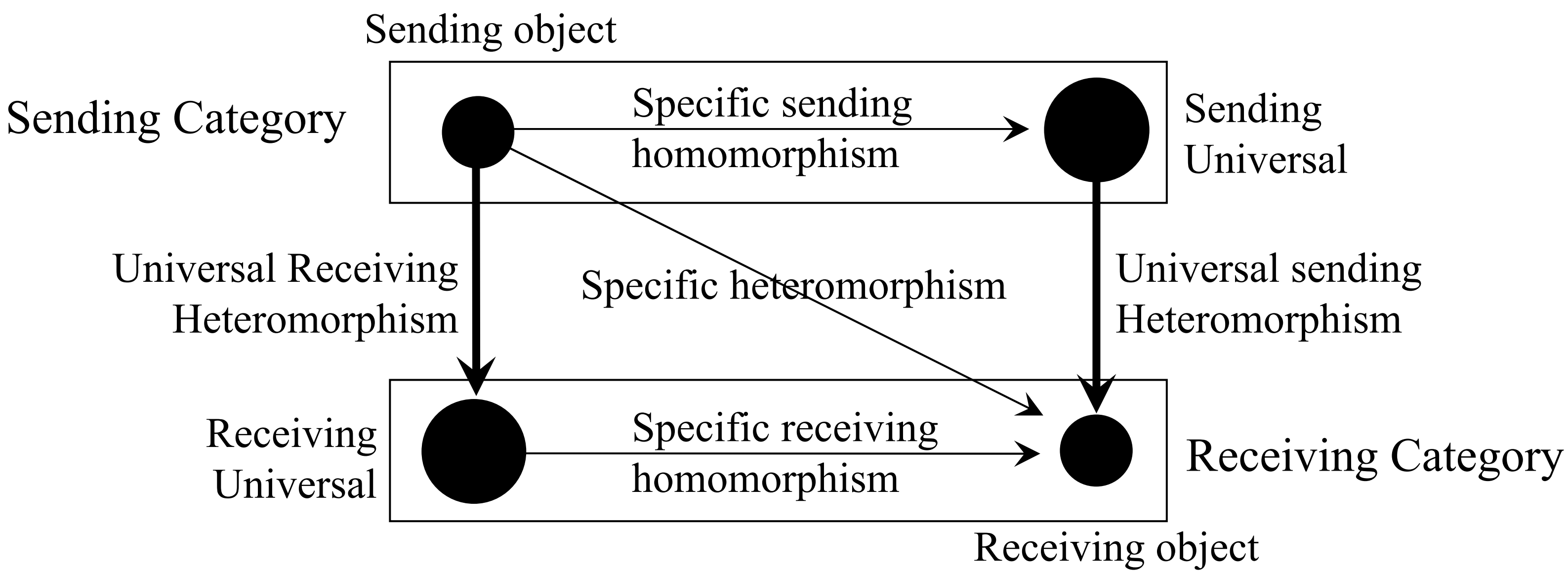
Right Half-adjunction: “Action”



Right Half-adjunction (universal in sending category)

- “Action” side is dual to “recognition” side.
- Category theory duality (turn the arrows around) is math version of sensory-motor, afferent-efferent, perception-behavior duality.

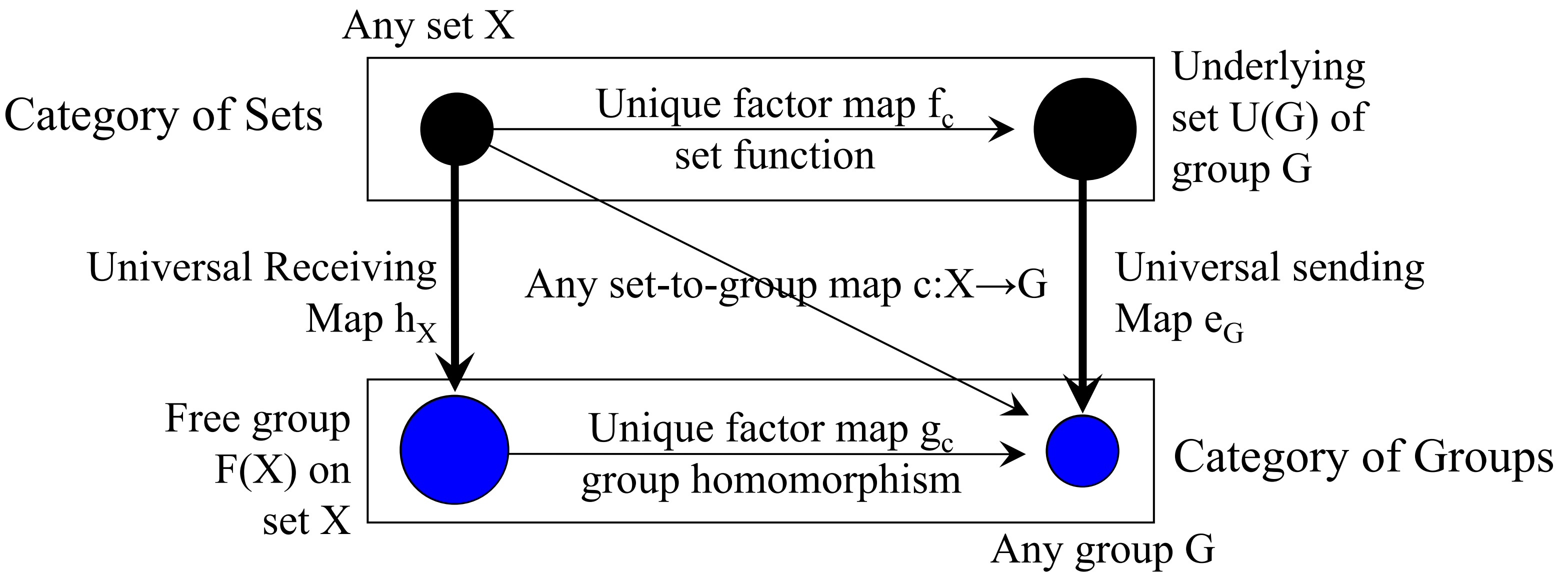
Adjunction = Left + Right Half-adjunctions



An Adjunction

- Left adjoint functor takes sending object to receiving universal;
- Right adjoint functor takes receiving object to sending universal.

Example: Free Group-Underlying Set Adjunction



Free Group-Underlying Set Adjunction

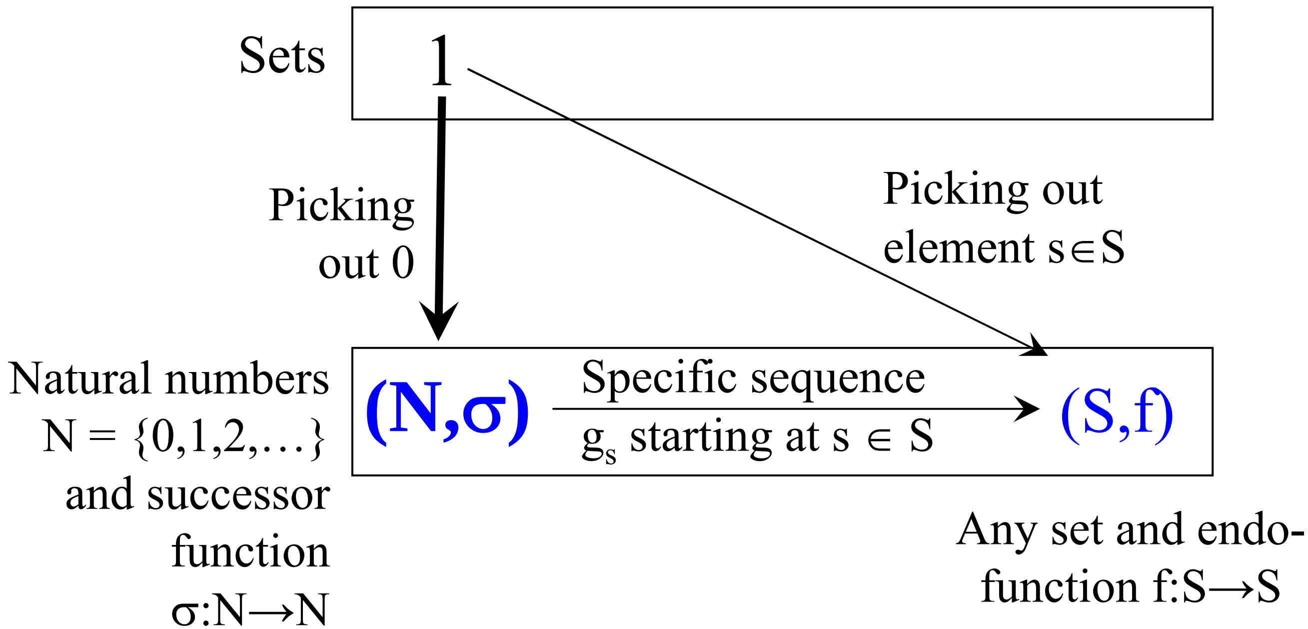
- $Hom_{Set}(X,U(G))$ = {set functions $f_c:X\to U(G)$};
- $Hom_{Group}(F(X),G)$ = {group homomorphisms $g_c:F(X)\to G$};
- Het(X,G) = {set-to-group maps $c:X\to G$;
- Basic isomorphisms of an adjunction:

$$Hom_{Group}(F(X),G) \cong Het(X,G) \cong Hom_{Set}(X,U(G)), \text{ i.e., } g_c \leftrightarrow c \leftrightarrow f_c.$$

Chomskyan Properties in Free Group Example

- Rich profligate internal structure: Free group $F(X)$ combinatorially generates group structure, i.e., all possible group elements, from the set of generators X.
- Impoverished inputs: Only the definition of g_c on the generators X is needed to uniquely determine the whole group homomorphism $g_c:F(X)\to G$.
- Active role of internal mechanism: Rather than just having receiver end of $c:X\to G$ (like stamp in wax G), internalized version of whole map is generated as $g_c:F(X)\to G$.
- Relative autonomy of internal mechanism: Minimal input of g_c on generators X gives internally generated 'message' $g_c:F(X)\to G$.
- Nota bene: the underlying set functor $U(G)$ was *not* mentioned above; only the left half-adjunction $F(X)$ was used. In most adjunctions, only one of the half-adjunction is interesting (the other half just fills out the adjunction).

Recursion as a left half-adjunction



- Universal object = Natural numbers with successor function $\sigma(0) = 1$, $\sigma(1) = 2$,…
- External map s from set 1={*} to S picks out any element $s \in S$.
- There exists unique factor map $g_s:N\rightarrow S$ such that $g_s(0) = s$ and for any $n \in N$, $g_s(\sigma(n)) = f(g_s(n))$ so g_s enumerates sequence: $g_s(0) = s$, $g_s(1) = f(s)$, $g_s(2) = f(f(s))$,…, $g_s(n) = f^n(s)$,....
- This universal mapping property is equivalent to the Peano Axioms that characterize the natural numbers N.

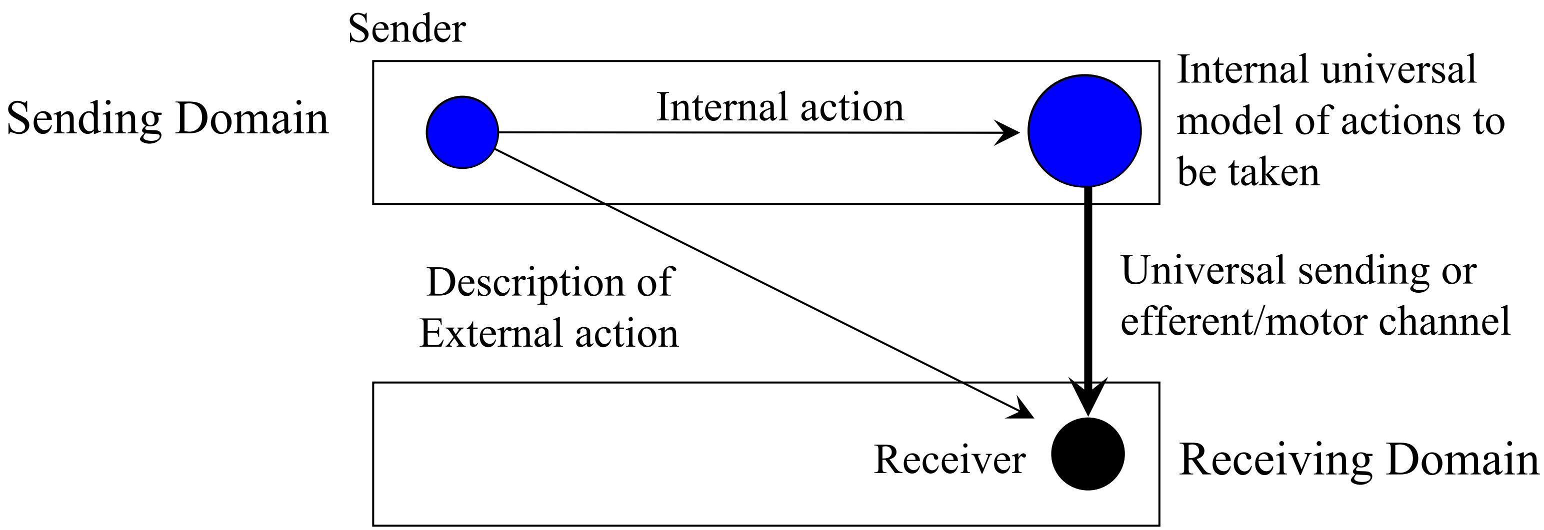
Properties in recursion example

- Rich internal structure: N is freely generated by starting with one element 0 and repeated applying the successor function σ that just generates a new element each time.
- Poverty of inputs: The map $g_s : N \rightarrow S$ is determined on the infinite values of N by the one value $g_s(0) = s$, i.e., an "infinite use of finite means."
- Active role of internal mechanism: universal object N generated from 0 by σ independent of the sending or 'environment' category of sets.

Dual Selectionist/Universalist Mechanisms

- In (ordinary) selectionist case, environment = sender, organism = receiver, and external map c internalized as internal-to-receiver map g_c, a "perception".
- In dual selectionist case, organism = sender, environment = receiver, and internal-to-sender map f_c, a "action", is externalized as external map c.
- Dual to everyday examples:
  * Tailor (now the sender) executes "42 long…" action by making such a suit for customer (now the receiver).
  * Telegraph operator delivers "Message #6" by communicating "Congratulations on the birth of your daughter" to recipient.

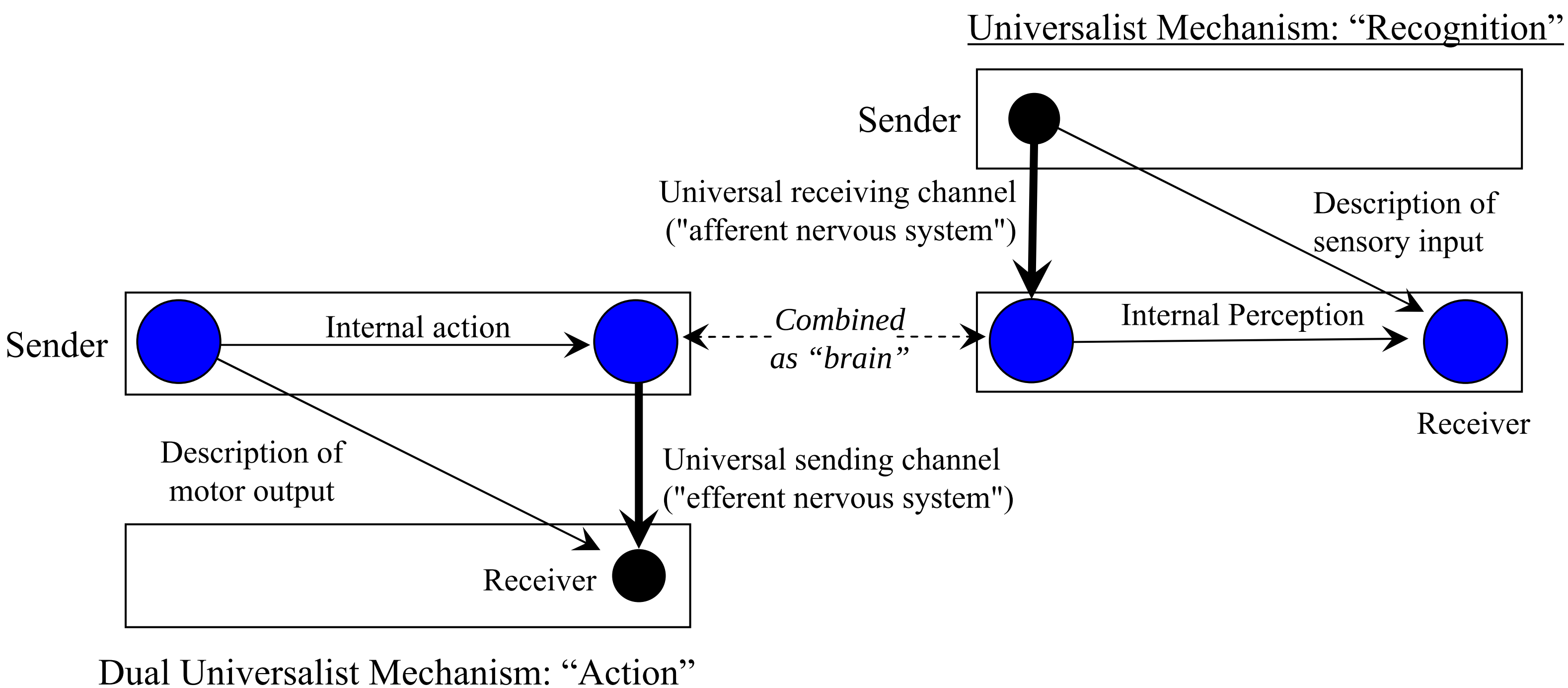
Dual Case as Universalist "Action"



Dual Mechanism for Selectionist "Action"

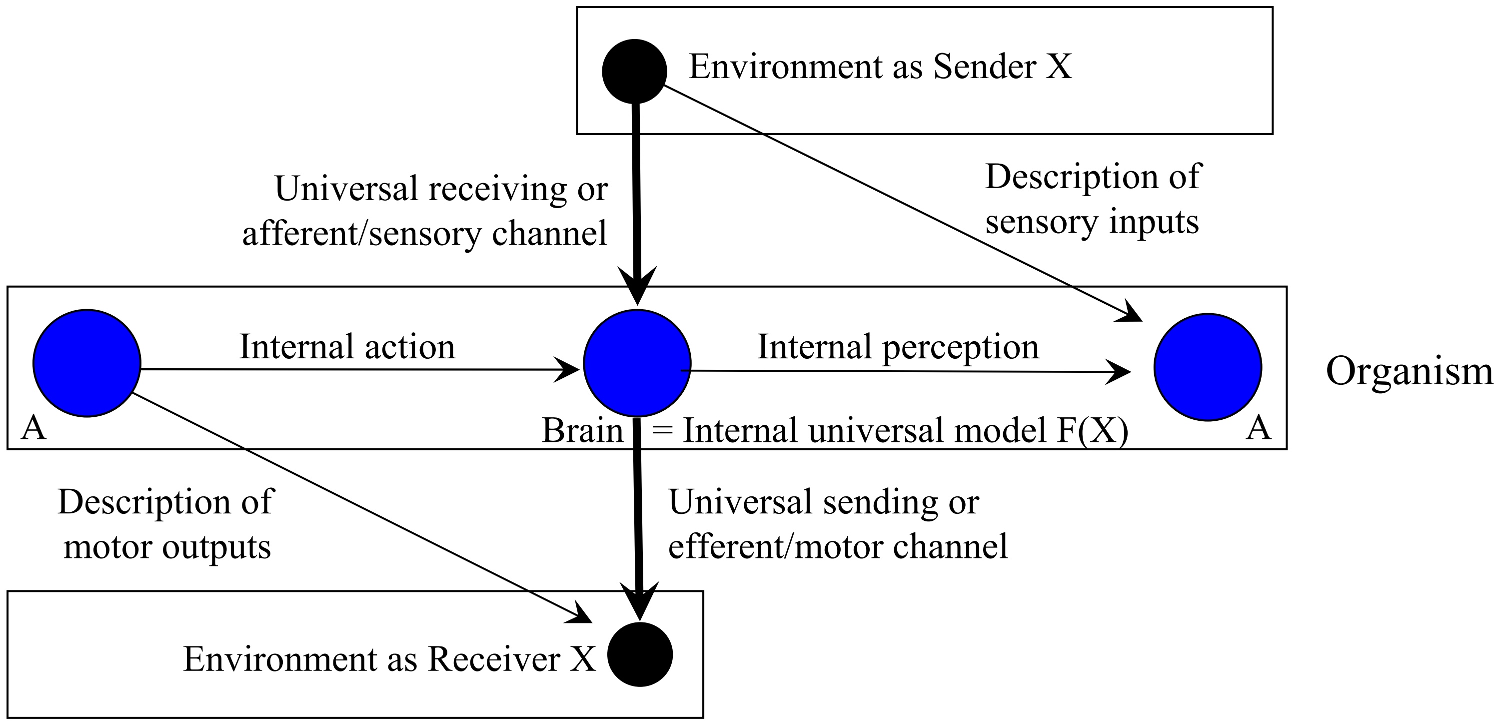
- Example of language speaking:
  - * External action = generating auditory signals, e.g., "Dicto ergo sum" (could be person speaking or tape recorder insofar as external signal is concerned);
  - * Internal action = speech act of saying "Dicto ergo sum" from potentially infinite repertoire of language faculty;
  - * Externalized through universal efferent channel as auditory signals "Dicto ergo sum".

Recognition + Action = Brain



- Adjoint functors combine two half-adjunctions in one way; let’s combine two half-adjunctions in the other way.
- Since the relevant parts in the “recognition” and “action” mechanisms are two half-adjunctions, let’s put the two cases together so the *same* universal model can do *both* “recognition” and “action”. Such a universal model is a model of a (human) “brain” .

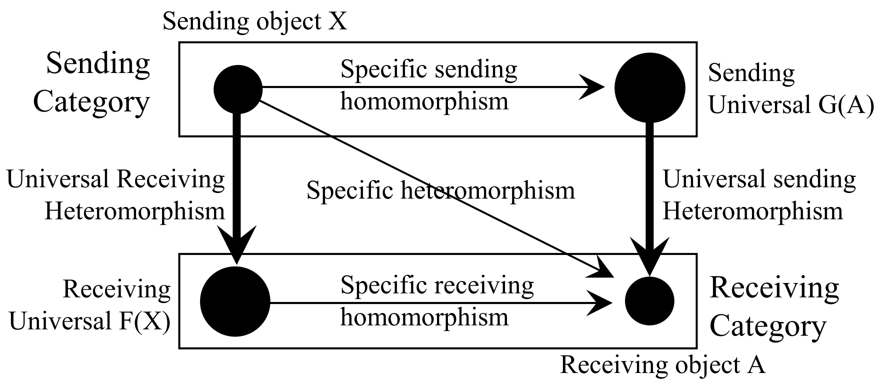
"Brain" as Functor



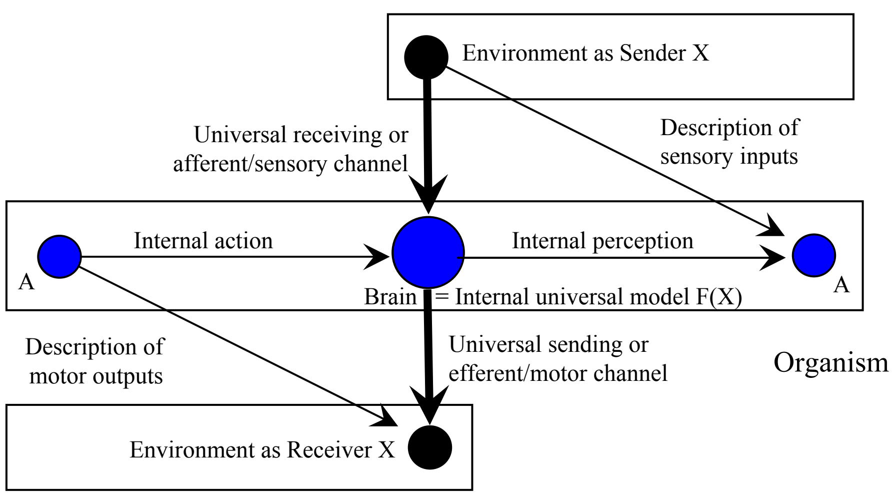
<u>Brain = functor giving left and right half-adjunction</u>

- A *brain functor* is a functor that represents heteromorphisms both ways between two categories (think "organism" & "environment"), e.g., any functor with left and right adjonts.
- External sensory input ↔ Internal perception, i.e., $\mathrm{Het}(X,A) \cong \mathrm{Hom}_{\mathrm{Organism}}(F(X),A)$;
- External motor output ↔ Internal action, i.e., $\mathrm{Het}(A,X) \cong \mathrm{Hom}_{\mathrm{Organism}}(A,F(X))$.

Adjoints & Brain functors as cognate concepts



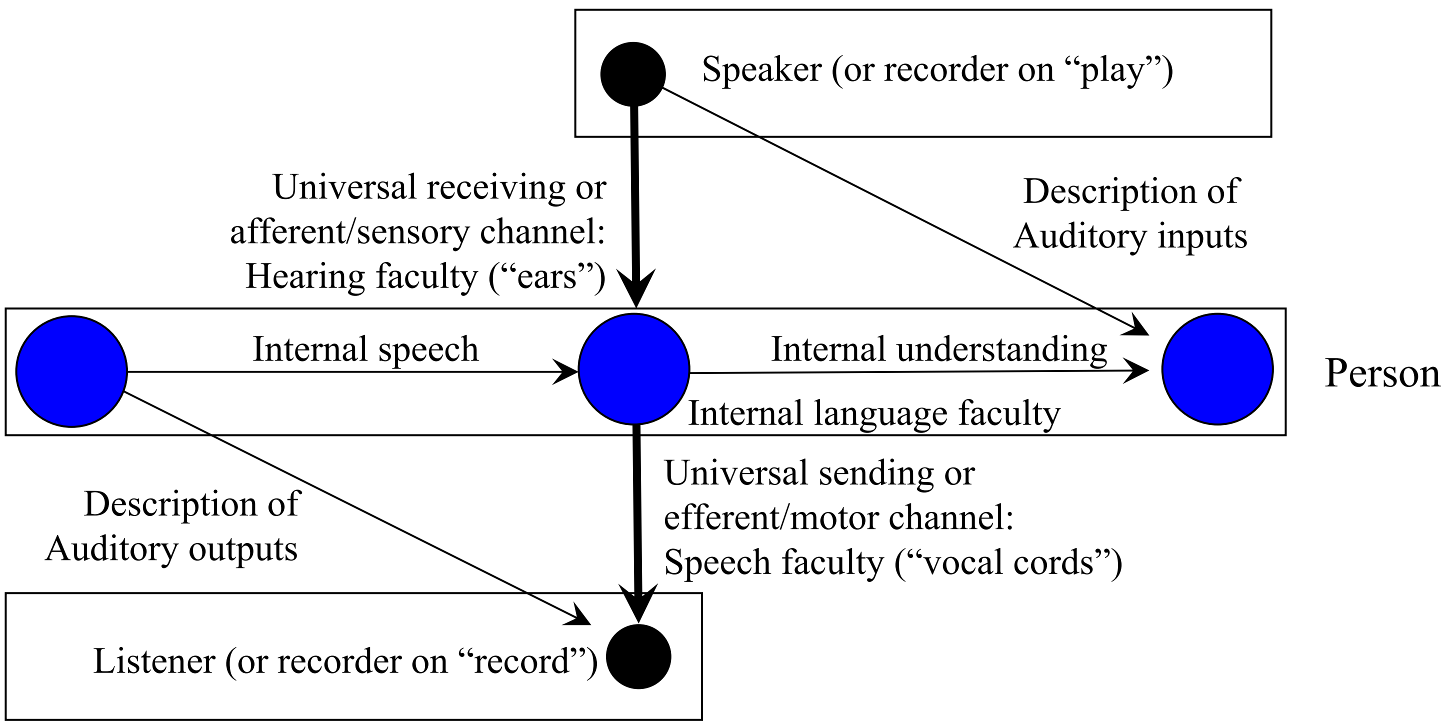
A Pair of Adjoint Functors



A Brain functor

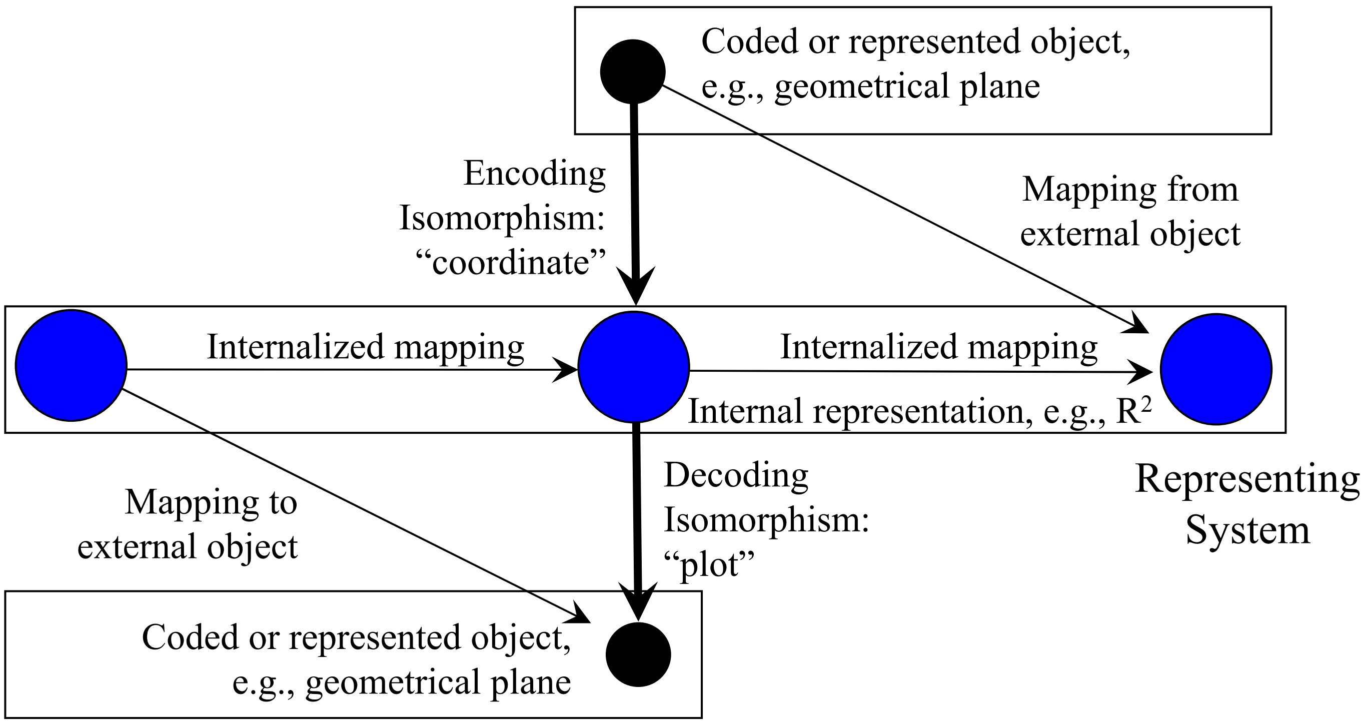
- A *left adjoint* F represents on the left the hets going one-way between two categories, Hom(F(X),A) ≅ Het(X,A); a *right adjoint* G represents on the right the hets going the same way between the two categories, Het(X,A) ≅ Hom(X,G(A)).
- A *brain functor* F is one functor that represents on the left the hets going one way, Hom(F(X),A) ≅ Het(X,A), and represents on the right the hets going the other way, Het(A,X) ≅ Hom(A,F(X)), between two categories.

Example Application: Language Use



<u>Brain as faculty for language understanding and speech</u>

Simple Encoding-Decoding Scheme

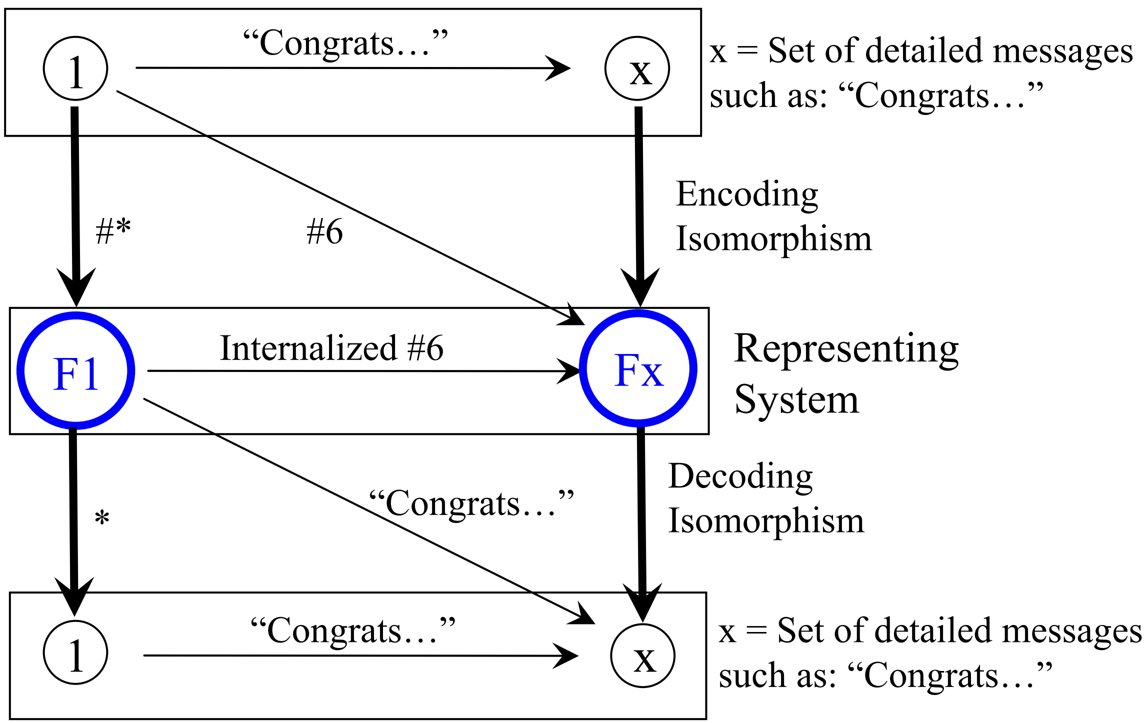


Simple Encoding-Decoding Scheme

- Simplest “brain” function is as a representational system to encode from the environment and to decode to the environment.[23]

Encoding-Decoding with Message Example

- x = {set of detailed messages, e.g., "Congrats…"};
- Fx = {set of x-message codes, e.g., #6};
- 1 = {*}, generic singleton set,
- F1 = {#*}; generic singleton code.
- In top square (lower triangle),"Internalized #6" is a recognition.
- In bottom square (top triangle), "Internalized #6" is an acton.



Encoding-Decoding Scheme Applied to Message Example

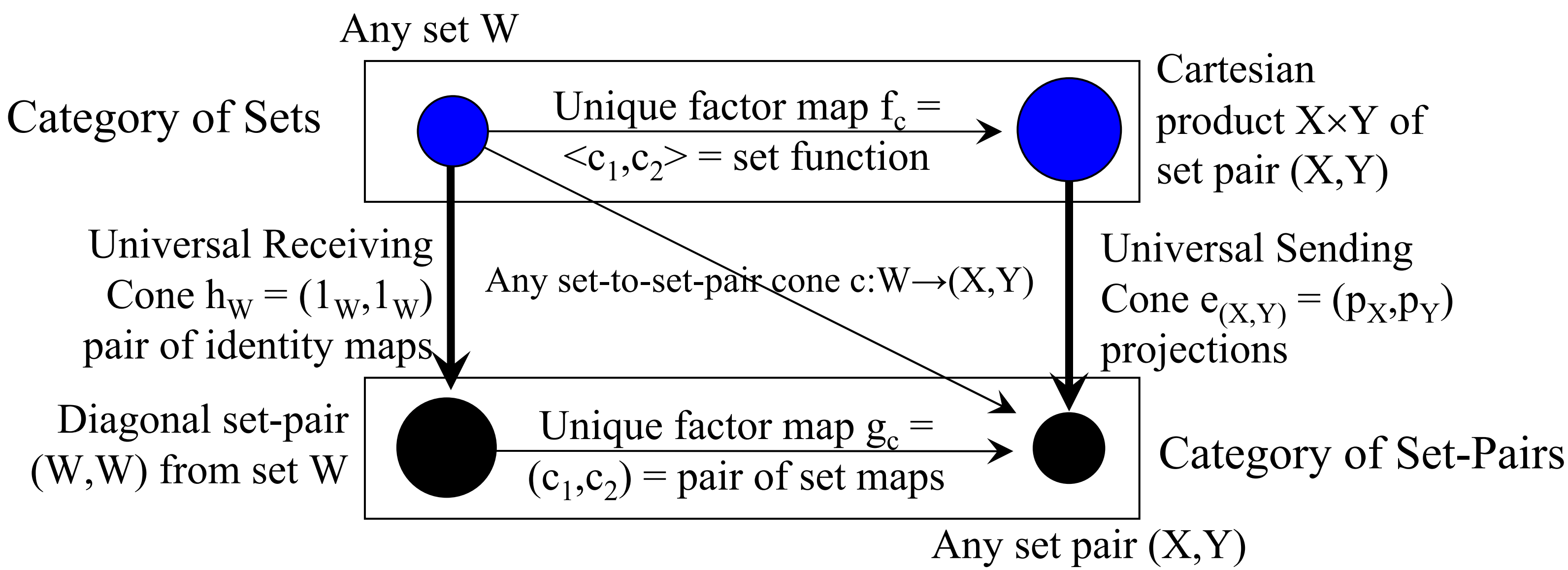
Conclusions

- General universalist mechanism "Recognition" has idealized math model as a left half-adjunction. Dual mechanism "Action" has idealized math model as a right half-adjunction.
- When *same* functor gives left half-adj. for "recognition" and right half-adj. for "action", then it is an idealized math model for a "brain" (with two univ. maps as "afferent" & "efferent" nervous systems).
- Key math concept is "universal mapping properties" (UMPs) or half-adjunctions which are building blocks of two related or cognate concepts: adjoint functors and "brain" functors.
- Basic ideas of Chomskyan themes: Universal model has *capacity* to generate 'all possible structures' according to given rules; specific structure generated by impoverished inputs.
- Some "selectionist" models have only "selectionist" talk with no universal models (e.g., reinforcement in behaviorism as "selection").
- *The point* is the conceptual structure (UMPs), isolated and characterized by category theory, to model Chomskyan themes.

Appendix: Other Math Examples

- Cartesian product as a dual selectionist model.
- Encoding and decoding of points in a plane.
- Inverse image $f^{-1}(\ )$ as a brain functor.
- Biproduct as "brain".

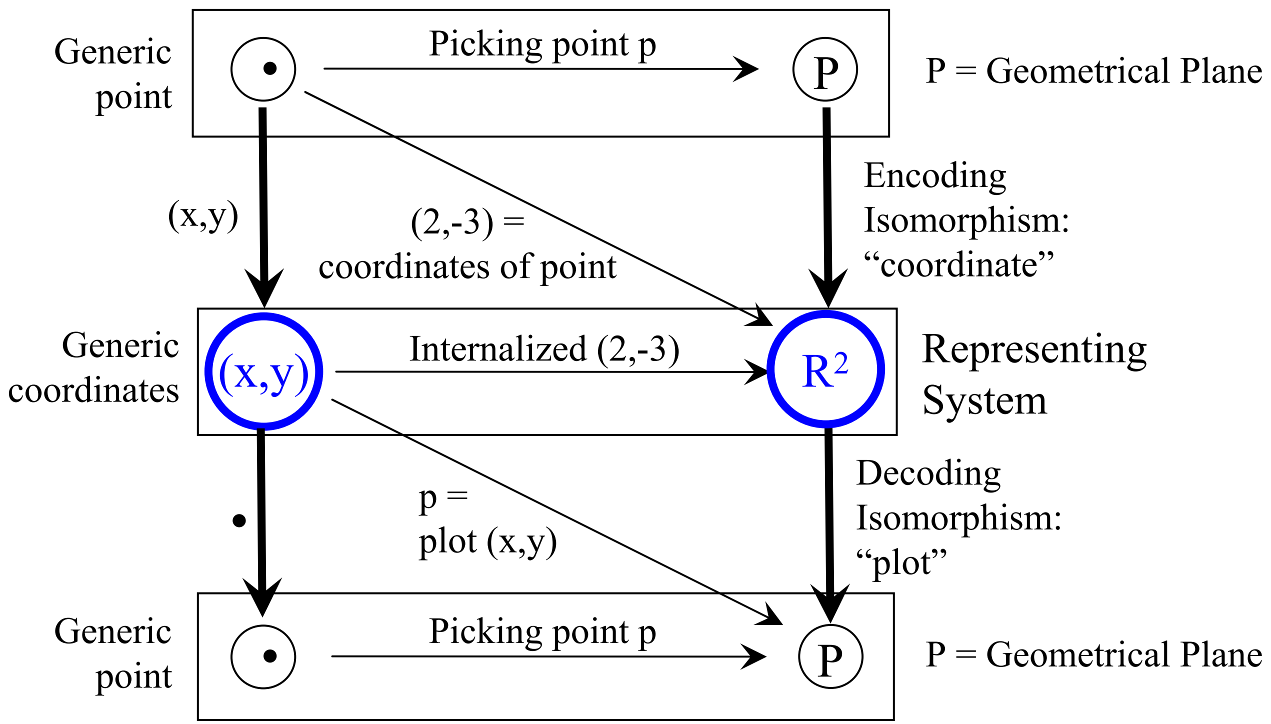
Cartesian Products: A Dual Example



Cartesian Product Adjunction

- In this adjunction, the right half-adjunction (northeast triangle) is the non-trivial part.
- A set-to-set-pair map is a "cone" $c = (c_1,c_2){:}W\rightarrow(X,Y)$ of two maps $c_1{:}W\rightarrow X$ and $c_2{:}W\rightarrow Y$ with same source W. Each element $w\in W$ maps to a pair of elements $c_1(w)\in X$ and $c_2(w)\in Y$.
- Universal object X×Y is constructed from all possible pairs (x,y) where $x\in X$ and $y\in Y$.
- One-to-one correspondence between external cones $c{:}W\rightarrow(X,Y)$ and internal factor maps $\langle c_1,c_2\rangle{:}W\rightarrow X\times Y$.

Encoding and Decoding of Points in Plane

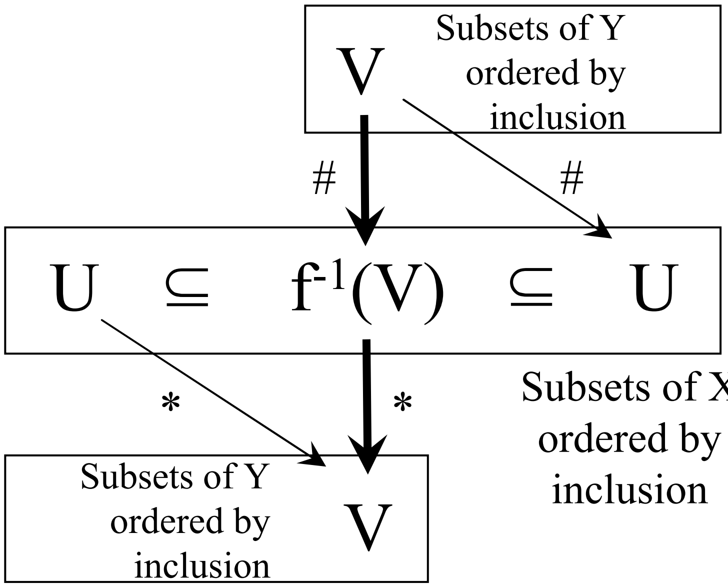


Encoding-Decoding Scheme
Applied to Points in the Plane

Inverse Image $f^{-1}()$ as a Brain Functor

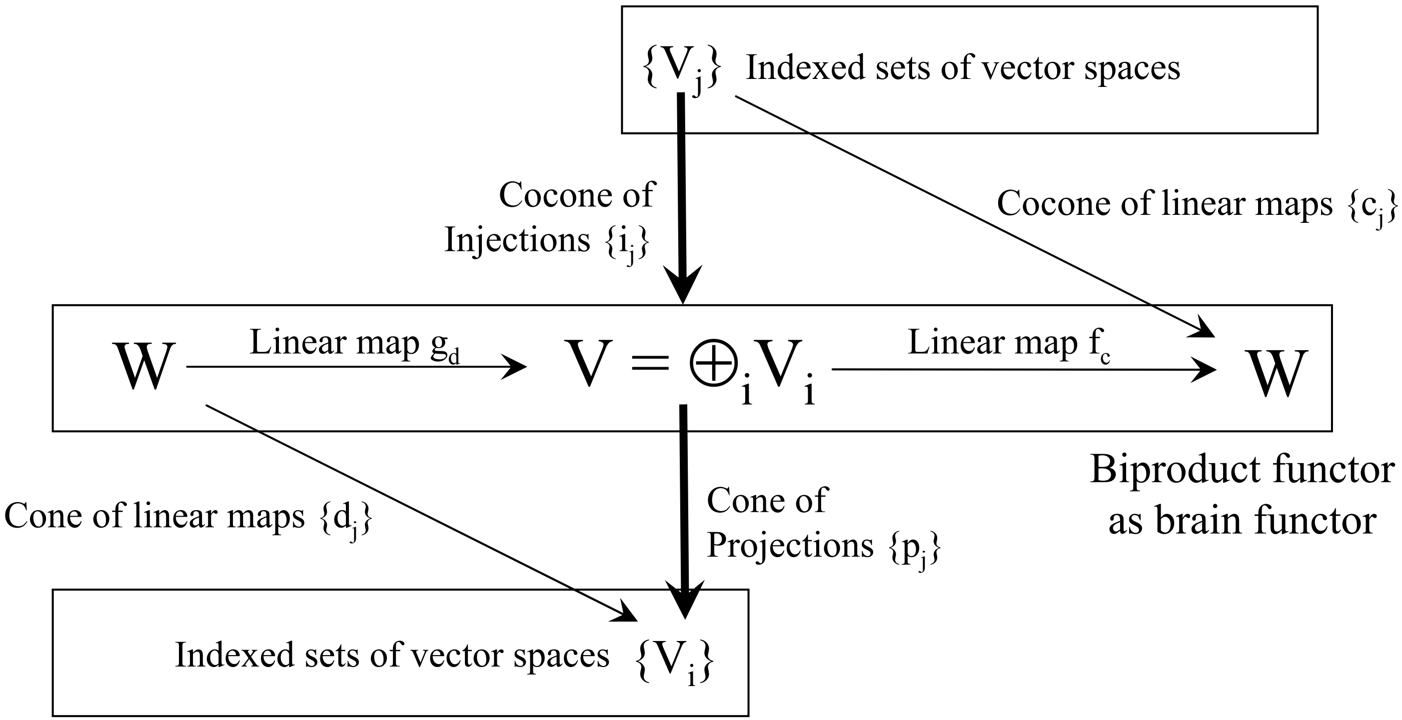
- Basic data: function $f:X \to Y$ from a set X to a set Y.
- "Environment" = Subsets V of Y ordered by inclusion.
- "Organism" = Subsets U of X ordered by inclusion.
- External input relation = V#U = all points of X that map by f into V are in U.
- $V \# f^{-1}(V)$, and $f^{-1}(V)$ = minimal U such that V#U so that: V#U if and only if $f^{-1}(V) \subseteq U$.
- External output relation = U*V = All points of U map by f into V.
- $f^{-1}(V) * V$, and $f^{-1}(V)$ = maximal U such that U*V so that: U*V if and only if $U \subseteq f^{-1}(V)$.

- Internal relations = $\subseteq$ (inclusion).
- Brain functor conditions for $f^{-1}(\ )$:
  - * V#U if and only if $f^{-1}(V) \subseteq U$, i.e.,
  - * External input relation # is internalized by internal "perception" $\subseteq$; and
  - * $U \subseteq f^{-1}(V)$ if and only if U*V, i.e.,
  - * Internal "action" $\subseteq$ is externalized by external output relation *.



<u>Inverse image as brain functor</u>

Biproduct as "brain"



Biproduct functor as brain functor

- Top & bottom category = category of sets of vector spaces over some field K indexed by finite set J.
- Middle category = category of vector spaces over K.
- Biproduct $\oplus_j V_j$ is both the product and coproduct of the vector spaces $\{V_j\}$.