

From Abstract Entities in Mathematics to Superposition States in Quantum Mechanics

David Ellerman
University of California at Riverside

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Abstract

Given an equivalence relation \sim on a set U , there are two abstract notions of an *element* of the quotient set U/\sim . The #1 abstract notion is a set $S = [u]$ of equivalent elements of U (an equivalence class); the #2 notion is an abstract entity u_S that is definite on what is common to the elements of the equivalence class S but is otherwise indefinite on the differences between those elements. For instance, the #1 interpretation of a homotopy type is an equivalence class of homotopic spaces, but the #2 interpretation, e.g., as developed in homotopy type theory, is an abstract space (without points) that has the properties that are in common to the spaces in the equivalence class but is otherwise indefinite. In philosophy, the #2 abstract entities might

be called *paradigm-universals*, e.g., ‘*the white thing*’ as opposed to the #1 abstract notion of "the set of white things" (out of some given collection U).

The paper shows how this #2 notion of a paradigm may be mathematically modeled using incidence matrices in Boolean logic and density matrices in probability theory. Then we cross the bridge to the density matrix treatment of the indefinite superposition states in quantum mechanics (QM). This connection between the #2 abstracts in mathematics and ontic indefinite states in QM elucidates Abner Shimony’s literal or objective indefiniteness interpretation of QM.

1 Introduction

The purpose of this paper is to illuminate the late Abner Shimony’s concept of an objectively indefinite superposition of definite eigen-states in quantum mechanics.

From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. [10, p. 47]

The superposition vectors of QM should be seen as representing actual ontic indefinites, i.e., objectively indefinite states (not just epistemic indefinite states of knowledge).

In addition to the Shimony’s phrase "objective indefiniteness," other philosophers of physics have used similar phrases for these indefinite states:

- Peter Mittelstaedt’s "incompletely determined" quantum states with "objective indeterminateness" [9];
- Paul Feyerabend’s "inherent indefiniteness" [5];
- Allen Stairs’ "value indefiniteness" and "disjunctive facts" [12];
- Steven French and Decio Krause’s "ontic vagueness" [6]; or
- E. J. Lowe’s "vague identity" and "indeterminacy" that is "ontic" [8].

But how can we understand the notion of an "ontic indefinite state"?

2 Two Versions of Abstraction

The claim is that we already have the notion of an indefinite state in the mathematical notion of an entity that abstracts as definite what is common to the distinct elements of a set S and rendering their differences as indefinite.

Given an equivalence relation on a set U such as "having the same color" and if $u \sim u'$ were white, then there are two notions of abstraction:

1. the #1 version of the abstraction operation takes equivalent entities $u \sim u'$ to the equivalence class $[u] = [u']$ of all white entities (in some universe U), and;
2. the #2 version of the abstraction operation takes all the equivalent entities $u \sim u'$ to the abstract entity "*the white entity*" that is definite on what is common in the set of all particular white things but is indefinite on how they differ (e.g., on all the other properties that distinguish them).

For instance, there are two notions of an ‘element’ of a quotient set or a quotient group (or any other quotient object in algebra):

1. a quotient group element as an equivalence class or coset; or
2. a quotient group element as an abstract entity representing what is common to the equivalence class.

Given *any* property $S(u)$ defined on the elements of U , two abstract objects can be defined:

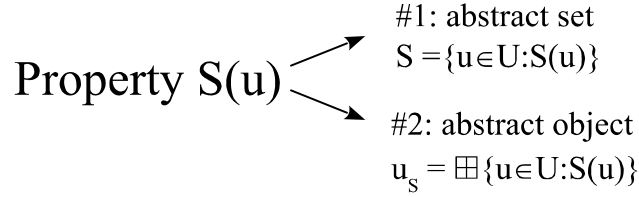


Figure 1: A property determines two types of abstract objects.

Intuitively the #2 abstract object u_s is ‘the paradigm S -entity’ (the blob-sum \boxplus is defined below) which is definite on the $S(u)$ property and indefinite on (i.e., blobs out) the differences between all the $u \in U$ such that $S(u)$.

3 An Example Starting with Attributes

Consider three predicates (binary attributes) $P(x)$, $Q(x)$, and $R(x)$ which could distinguish at most $2^3 = 8$ definite-particular entities: u_1, \dots, u_8 called *eigen-elements* and which can be presented in a table like a truth table:

$P(x)$	$Q(x)$	$R(x)$	u
1	1	1	u_1
1	1	0	u_2
1	0	1	u_3
1	0	0	u_4
0	1	1	u_5
0	1	0	u_6
0	0	1	u_7
0	0	0	u_8

Table 1: Eight entities specified by 3 properties.

The general rule is if $f, g, h : U \rightarrow \mathbb{R}$ are numerical attributes with the number of distinct values as n_f, n_g , and n_h respectively, then those attributes could distinguish or classify $n_f \times n_g \times n_h$ distinct subsets of U . If the join of the inverse-image partitions is the discrete partition, i.e., $\{f^{-1}\} \vee \{g^{-1}\} \vee \{h^{-1}\} = \mathbf{1}_U$ [3], then $\{f, g, h\}$ is a *complete set of attributes* since they can distinguish or classify the eigen-elements of U . Then we can distinguish the elements of U by their triple of values, i.e., $|f(u_j), g(u_j), h(u_j)\rangle$ uniquely determines $u_j \in U$.

In the example, any subset $S \subseteq U = \{u_1, \dots, u_8\}$ is characterized by a property $S(x)$, the *disjunctive normal form property*, common to all and only the elements of S . If $S = \{u_1, u_4, u_7\}$, then the DNF property is:

$$S(x) = [P(x) \wedge Q(x) \wedge R(x)] \vee [P(x) \wedge \neg Q(x) \wedge \neg R(x)] \vee [\neg P(x) \wedge \neg Q(x) \wedge R(x)].$$

But what are the #1 and #2 abstract entities?

1. The #1 abstract entity is the set

$$S = \{u_i \in U | S(u_i)\} = \{u_1, u_4, u_7\}$$

of all the distinct $S(x)$ -entities; and

2. The #2 abstract entity is *the paradigm-universal* $S(x)$ -entity symbolized

$$u_S = u_1 \boxplus u_4 \boxplus u_7 = \boxplus \{u_i \in U | S(u_i)\}$$

The ‘superposition’ or ‘blob-sum’ of u_1 , u_4 , and u_7 .

that is *definite on the DNF property* $S(x)$ but indefinite on what distinguishes the different $S(x)$ -entities. Thus $S(u_S)$ holds but none of the disjuncts hold since that would make u_S equal to u_1 , u_4 , or u_7 . Hence $S(u_S)$ is a ‘disjunctive fact’ in the sense of Allen Stairs [12].

4 Some Philosophical Concerns

It is best to think of S as the set of *definite particular* $S(x)$ -entities in some universe U , while u_S is the *indefinite paradigm-universal* $S(x)$ -entity is the ‘superposition’ $u_S = \boxplus \{u_i \in U | S(u_i)\}$ that is, in general, "one *over* the many." Only when $S = \{u_j\}$ is a singleton does the definite description ‘*the S-entity*’ refer to an element of U , i.e., $u_{\{u_j\}} = u_j$.

Making the "one" $u_S = \boxplus \{u_i \in U | S(u_i)\}$ *over* the many, i.e., more abstract than the $u_i \in U$ (for $|S| > 1$) avoids the paradoxes just as the iterative notion of set does in ordinary set theory, i.e., for #1 type of abstractions. Otherwise, if we ignore the given set U , then we can recreate Russell’s Paradox for $R(u_S) \equiv \neg S(u_S)$ so:

$$u_R = \boxplus \{u_S | \neg S(u_S)\} \text{ and thus } R(u_R) \text{ implies } \neg R(u_R), \text{ and } \neg R(u_R) \text{ implies } R(u_R).$$

But if we define $u_R = \boxplus \{u_S \in U | \neg S(u_S)\}$, then assuming $u_R \in U$ leads to the contradiction so $u_R \notin U$.

The paradigm-universal u_S is *not* universal ‘ S -ness’. Where $S(x)$ is being white, then $u_{white} =$ ‘*the white thing*’, not ‘whiteness’. This distinction goes back to Plato:

But Plato also used language which suggests not only that the Forms exist separately ($\chi\omega\rho\iota\sigma\tau\alpha$) from all the particulars, but also that each Form is a peculiarly accurate or good particular of its own kind, i.e., the standard particular of the kind in question or the model ($\pi\alpha\rho\rho\alpha\delta\epsilon\iota\gamma\mu\alpha$) to which other particulars approximate. [7, p. 19]

Some have considered interpreting the Form as *paradeigma* as an error.

For general characters are not characterized by themselves: humanity is not human. The mistake is encouraged by the fact that in Greek the same phrase may signify both the concrete and the abstract, e.g. $\lambda\epsilon\nu\kappa\omicron\nu$ (literally "the white") both "the white thing" and "whiteness", so that it is doubtful whether $\alpha\nu\tau\omicron\ \tau\omicron\ \lambda\epsilon\nu\kappa\omicron\nu$ (literally "the white itself") means "the superlatively white thing" or "whiteness in abstraction". [7, pp. 19-20]

Thus for the abstract property $W(u)$ "whiteness", we have:

1. the #1 abstraction is *the set of white things* $W = \{u \in U : W(u)\}$, and;
2. the #2 abstraction ‘*the white thing*’ u_W .

5 Relations Between #1 and #2 Universals

For properties $S()$ defined on U , there is a 1-1 correspondence between the #1 and #2 universals:

$$\cup \{\{u\} | u \in U \& S(u)\} = S \longleftrightarrow u_S = \boxplus \{u_{\{u\}} | u \in U \& S(u)\}.$$

In each case, we may extend the definition of the property to the two universals. For $T()$ another property defined on U :

$$S(T) \text{ iff } (\forall u \in U) (T(u) \Rightarrow S(u)) \text{ iff } S(u_T).$$

In terms of the #1 universals, $S(S)$ holds by definition and: $S(T)$ iff $T \subseteq S$, and similarly $S(u_S)$ always holds. But what is the #2 universals equivalent of $T \subseteq S$? Intuitively u_S is ‘the S -thing’ that is definite on having the S -property but is otherwise indefinite on the differences between the members of S . If we make more properties definite, then in terms of subsets, that will in general cut down to a subset $T \subseteq S$, so u_T would inherit the paradigmatic property holding on the superset S , i.e., $S(u_T)$.

This "process" to changing to a more definite universal $u_S \rightsquigarrow u_T$ for $T \subseteq S$ will be called *projection* and symbolized:

$$u_T \triangleleft u_S \text{ (or } u_S \triangleright u_T)$$

u_T is a "sharpening" or more definite version of u_S .

$S()$ defined on U	#1 abstraction	#2 abstraction
Universals for $S()$	$S = \cup \{\{u\} u \in U \& S(u)\}$	$u_S = \boxplus \{u_{\{u\}} u \in U \& S(u)\}$
$T()$ defined on U	$S(T)$ iff $T \subseteq S$	iff $S(u_T)$ iff $u_T \triangleleft u_S$

Table 2: Equivalents between #1 and #2 universals

In the language of Plato, the projection relation \triangleleft is the relation of "participation" ($\mu\epsilon\theta\epsilon\xi\iota\varsigma$ or *methexis*). As Plato would say, u_T has the property $S()$ iff it participates in ‘the S -thing’, i.e., $S(u_T)$ iff $u_T \triangleleft u_S$.

Thus there are two theories of abstract objects:

1. Set theory is the theory of #1 abstract objects, the sets S , where (taking \in as the participation relation), sets are *never* self-participating, i.e., $S \notin S$;
2. There is a second theory about the #2 abstract entities, the paradigms u_S , which are *always* self-participating, i.e., $u_S \triangleleft u_S$.

Like sets S , the #2 abstract entities u_S , the *paradigm-universals*, are routinely used in mathematics.

6 Examples of Abstract Paradigms in Mathematics

There is an equivalence relation $A \simeq B$ between topological spaces which is realized by a continuous map $f : A \rightarrow B$ such that there is an inverse $g : B \rightarrow A$ so the $fg : B \rightarrow B$ is homotopic to 1_B (i.e., can be continuously deformed in 1_B) and gf is homotopic to 1_A . Classically "Homotopy types are the equivalence classes of spaces" [2] under this equivalence relation. That is the #1 type of abstraction.

But the interpretation offered in homotopy type theory is expanding identity to "coincide with the (unchanged) notion of equivalence" [13, p. 5] so it would refer to the #2 homotopy type, i.e., ‘the homotopy type’ that captures the mathematical properties shared by all spaces in an equivalence class of homotopic spaces (wiping out the differences). Note that ‘the homotopy type’ is not one of the classical topological spaces (with points etc.) in the #1 equivalence class of homotopic spaces.

While classical homotopy theory is analytic (spaces and paths are made of points), homotopy type theory is synthetic: points, paths, and paths between paths are basic, indivisible, primitive notions. [13, p. 59]

Homotopy type theory systematically develops a theory of the #2 type of abstractions that grows out of homotopy theory and type theory in a new foundational theory.

From the logical point of view, however, it is a radically new idea: it says that isomorphic things can be identified! Mathematicians are of course used to identifying isomorphic structures in practice, but they generally do so by “abuse of notation”, or some other informal device, knowing that the objects involved are not “really” identical. But in this new foundational scheme, such structures can be formally identified, in the logical sense that every property or construction involving one also applies to the other. [13, p. 5]

Our purpose is rather more modest, to model the theory of paradigm-universals u_S and their projections u_T —that is analogous to working with sets and subsets, e.g., in a Boolean algebra of subsets. That is all we will need to show that probability theory can be developed using paradigms u_S instead of subset-events S , and to make the connection to quantum mechanics.

Another homotopy example is ‘the path going once (clockwise) around the hole’ in an annulus A (disk with one hole), an abstract entity $1 \in \pi_0(A) \cong \mathbb{Z}$:

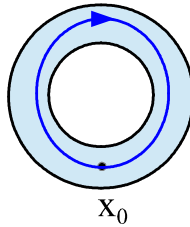


Figure 2: ‘the path going once (clockwise) around the hole’

Note that ‘the path going once (clockwise) around the hole’ has the paradigmatic property of “going once (clockwise) around the hole” but is *not* one of the particular (coordinatized) paths that constitute the equivalence class of coordinatized once-around paths deformable into one another.

In a similar manner, we can view other common #2 abstractions such as: ‘the cardinal number 5’ that captures what is common to the isomorphism class of all five-element sets; ‘the number 1 mod (n) ’ that captures what is common within the equivalence class $\{\dots, -2n + 1, -n + 1, 1, n + 1, 2n + 1, \dots\}$ of integers; ‘the circle’ or ‘the equilateral triangle’—and so forth.

Category theory helped to motivate homotopy type theory for good reason. Category theory has no notion of identity between objects, only isomorphism as ‘equivalence’ between objects. Therefore category theory can be seen as a theory of *abstract* #2 objects (“up to isomorphism”), e.g., abstract sets, groups, spaces, etc.

7 The Connection to Interpreting Symmetry Operations

The difference between the #1 abstract set and the #2 abstract entity can also be visually illustrated in a simple example of the symmetry operation (defining an equivalence relation) of reflection on the aA -axis for a fully definite isosceles triangles:

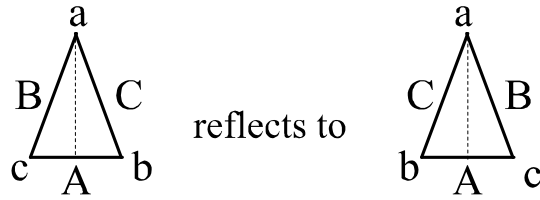


Figure 3: Reflection on vertical axis symmetry operation.

Thus the equivalence class of reflective-symmetric figures in the #1 or classical interpretation is the set:

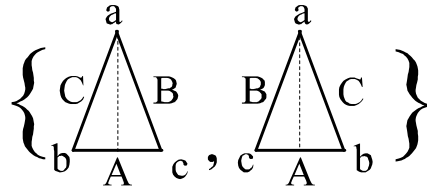


Figure 4: The #1 abstraction of equivalence class.

But under the #2 or indefiniteness-abstraction(-quantum) interpretation, the equivalence abstracts to the figure that is definite as to what is the same and indefinite as to what is different between the definite figures in the equivalence class:

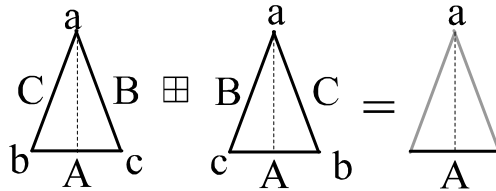


Figure 5: The #2 abstraction of indefinite entity.

Note that the symmetry operation on the indefinite figure is the *identity*. As noted in the discussion of homotopy type theory, the movement from the #1 equivalence class S to the #2 abstract-indefinite entity u_S replaces equivalence with identity. That is because the symmetry operation goes from one element in an equivalence class S to another element in S that differs in some definite aspects, but those are precisely the aspects that are removed in the indefinite-abstract u_S —so the symmetry just takes u_S to itself.

Since we are later going to relate the #2 entities to the indefinite states of quantum mechanics, the example suggests that while classically a symmetry operation is invariant on an equivalence class S (i.e., takes one definite element in the equivalence class S to another definite element in S), in the #2 quantum case, the symmetry operation on the indefinite entity u_S is the *identity*.

This is illustrated in the transition from the classical Maxwell-Boltzmann statistics to the quantum Bose-Einstein statistics. Suppose we have two particles of the same type which are classically indistinguishable so, following Weyl, we distinguish them as Mike and Ike. If each of the two particles could be in states A , B , or C , then the set of possible states is the set of nine ordered pairs $\{A, B, C\} \times \{A, B, C\}$. Applying the symmetry operation of permuting Mike and Ike, we have six equivalence classes.

Equivalence classes under permutation	M-B
$\{(A, B), (B, A)\}$	$\frac{2}{9}$
$\{(A, C), (C, A)\}$	$\frac{2}{9}$
$\{(B, C), (C, B)\}$	$\frac{2}{9}$
$\{(A, A)\}$	$\frac{1}{9}$
$\{(B, B)\}$	$\frac{1}{9}$
$\{(C, C)\}$	$\frac{1}{9}$

Table 3: Maxwell-Boltzmann distribution.

Since the primitive data are the ordered pairs, we assign the equal probabilities of $\frac{1}{9}$ to each pair which results in the Maxwell-Boltzmann distribution for the equivalence classes.

But in the quantum case, we don't have an equivalence class S of distinct ordered pairs like $\{(A, B), (B, A)\}$ under the symmetry; we have a single indefinite entity $u_{\{(A, B), (B, A)\}}$ where the symmetry operation is the *identity*. Since there are now only six primitive entities, we assign the equal probabilities of $\frac{1}{6}$ to each entity and obtain the Bose-Einstein distribution.

Six indefinite states	B-E
$u_{\{(A, B), (B, A)\}}$	$\frac{1}{6}$
$u_{\{(A, C), (C, A)\}}$	$\frac{1}{6}$
$u_{\{(B, C), (C, B)\}}$	$\frac{1}{6}$
$u_{\{(A, A)\}}$	$\frac{1}{6}$
$u_{\{(B, B)\}}$	$\frac{1}{6}$
$u_{\{(C, C)\}}$	$\frac{1}{6}$

Table 4: Bose-Einstein distribution.

Ruling out repeated states (i.e., the Pauli exclusion principle), there are only three primitive entities and that gives the Fermi-Dirac distribution.¹

Three possible indefinite states	F-D
$u_{\{(A, B), (B, A)\}}$	$\frac{1}{3}$
$u_{\{(A, C), (C, A)\}}$	$\frac{1}{3}$
$u_{\{(B, C), (C, B)\}}$	$\frac{1}{3}$

Table 5: Fermi-Dirac distribution.

8 How to Model the #1 and #2 Abstracts

There are simple but different models to distinguish the #1 and #2 interpretations for $S \subseteq U$ with a finite $U = \{u_1, \dots, u_n\}$ such as:

$$U = \{ \triangle, \blacksquare, \blacklozenge, \blackhexagon \}$$

Figure 6: Universe U of figures

¹For more of this pedagogical model of QM using sets (where the sets may be given the #2 abstraction u_S interpretation), see [4].

Ordinarily the set of solid figures $S = \{u_2, u_3, u_4\} \subseteq \{u_1, u_2, u_3, u_4\} = U$ would be represented by a one-dimensional column vector $|S\rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, but by using a *two*-dimensional matrix, we can *represent* the *two* #1 and #2 versions of S as two types of incidence matrices.

1. The #1 (classical) representation of S (i.e., set of S -things or set of solid figures) is the diagonal

$$\text{matrix } \text{In}(\Delta S) \text{ that lays the column vector } |S\rangle \text{ along the diagonal: } \text{In}(\Delta S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

representation of set S of distinct S -entities. $\text{In}(\Delta S)$ is the *incidence matrix* of the diagonal $\Delta S \subseteq U \times U$ whose entries are the values of the characteristic function $\chi_{\Delta S}(u_j, u_k) = \delta_{jk} \chi_S(u_j)$.

2. The #2 (quantum) representation of S (i.e., *the* S -thing) is the matrix $\text{In}(S \times S)$ that uses a 1 in the row j , column k cell to mean u_j and u_k are both in S : $\text{In}(S \times S) = |S\rangle(|S\rangle)^t =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \text{representation of one indistinct } S\text{-thing, 'the solid figure' } u_S = u_2 \boxplus u_3 \boxplus u_4.$$

$\text{In}(S \times S)$ is the *incidence matrix* of the product $S \times S \subseteq U \times U$ (instead of the diagonal ΔS) with the entries $\chi_{S \times S}(u_j, u_k)$.

Note that for singletons $S = \{u_j\}$, $\text{In}(\Delta S) = \text{In}(S \times S)$ as expected, and for $|S| > 1$, $\text{In}(\Delta S) \neq \text{In}(S \times S)$.

The two representations differ only in the off-diagonal entries. Think of the off-diagonal $\text{In}(S \times S)_{j,k} = 1$'s as equating, cohering, or 'blobbing' together u_j and u_k :

$$\text{In}(S \times S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \text{ says}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & u_2 \sim u_3 & u_2 \sim u_4 \\ 0 & u_3 \sim u_2 & 1 & u_3 \sim u_4 \\ 0 & u_4 \sim u_2 & u_4 \sim u_3 & 1 \end{bmatrix}.$$

We now can represent the blob-sum #2 operation on entities: $u_S = \boxplus \{u_i \in U | S(u_i)\}$ as the *blob-sum* \boxplus of the corresponding incidence matrices:

$$\text{In}(S \times S) = \boxplus_{u_i \in S} \text{In}(\{u_i\} \times \{u_i\})$$

where the *blob-sum* \boxplus is defined for $S_1, S_2 \subseteq U$ with $S = S_1 \cup S_2$:

$$\begin{aligned} \text{In}(S_1 \times S_1) \boxplus \text{In}(S_2 \times S_2) &:= \text{In}(S \times S) = \text{In}((S_1 \cup S_2) \times (S_1 \cup S_2)) \\ &= \text{In}(S_1 \times S_1 \cup S_2 \times S_2 \cup S_1 \times S_2 \cup S_2 \times S_1) \\ &= \text{In}(S_1 \times S_1) \vee \text{In}(S_2 \times S_2) \vee \text{In}(S_1 \times S_2) \vee \text{In}(S_2 \times S_1). \end{aligned}$$

Disjunction: $\text{In}(S_1 \times S_1) \vee \text{In}(S_2 \times S_2) \vee$ blobbing cross-terms.²

²The disjunction of incidence matrices is the usual entry-wise disjunction: $1 \vee 1 = 1 \vee 0 = 0 \vee 1 = 1$ and $0 \vee 0 = 0$, and similarly for conjunction.

For $S = \{u_2, u_4\}$, the blob-sum $u_S = u_2 \boxplus u_4$ is represented by:

$$\text{In}(\{u_2\} \times \{u_2\}) \boxplus \text{In}(\{u_4\} \times \{u_4\}) = \text{In}(S \times S)$$

where the blob-sum operation \boxplus means ‘blobbing-out’ the distinctions between entities in S (given by the cross-terms in $\{u_2, u_4\} \times \{u_2, u_4\}$):

$$\begin{aligned} \text{In}(S \times S) &= \text{In}(\{u_2\} \times \{u_2\}) \boxplus \text{In}(\{u_4\} \times \{u_4\}) \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boxplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \text{In}(\{u_2, u_4\} \times \{u_2, u_4\}) \\ &= \text{In}(\{u_2\} \times \{u_2\}) \vee \text{In}(\{u_4\} \times \{u_4\}) \vee \text{In}(\{u_2\} \times \{u_4\}) \vee \text{In}(\{u_4\} \times \{u_2\}) \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Due to the development of Boolean subset logic and set theory, we are perfectly comfortable with considering the $\#1$ abstractions of sets S of even concrete ur-elements like the set of entities on a table. The representatives $\text{In}(\Delta S)$ trivially form a BA isomorphic to the BA of subsets $\wp(U)$.

To better understand abstraction in mathematics and indefinite states in QM, we should become as comfortable with paradigms u_S as with sets S . The paradigms u_S for $S \in \wp(U)$ form a Boolean algebra isomorphic to $\wp(U)$ under the mapping: for any Boolean operation $S \# T$ for $S, T \in \wp(U)$, $u_{S \# T}$ is the paradigm represented by $\text{In}((S \# T) \times (S \# T))$.

- The union of subsets $S \cup T$ induces the operation on paradigms represented by $\text{In}((S \cup T) \times (S \cup T)) = \text{In}(S \times S) \boxplus \text{In}(T \times T)$, so the *union or join of paradigms* is the blob-sum $u_{S \cup T} = u_S \boxplus u_T$ (note as expected, for $T \subseteq S$, $u_S \boxplus u_T = u_S$);
- The *intersection or meet of paradigms* $u_S \wedge u_T = u_{S \cap T}$ is represented by $\text{In}(S \cap T \times S \cap T) = \text{In}(S \times S) \wedge \text{In}(T \times T)$ (note as expected, for $T \subseteq S$, $u_S \wedge u_T = u_T$);
- The *negation of a paradigm* $\neg u_S = u_{S^c}$ is represented by $\text{In}(S^c \times S^c) = \boxplus \{\text{In}(\{u\} \times \{u\}) \mid u \notin S\}$ (note as expected, $u_S \boxplus u_{S^c} = u_U$).

9 The Projection Operation: Making an indefinite entity more definite

Now suppose we classify or partition *all* the elements of U according to an attribute such as the parity of the number of sides, where a *partition* is a set of disjoint subsets (blocks) of U whose union is all of U . Let π be the partition of two *blocks* $O = \{\text{Odd}\} = \{u_1, u_3\}$ and $E = \{\text{Even}\} = \{u_2, u_4\}$.

The equivalence relation defined by π is $\text{indit}(\pi) = (O \times O) \cup (E \times E)$ [3] and the disjunction is:

$$\begin{aligned} \text{In}(O \times O) \vee \text{In}(E \times E) &= \text{In}(\text{indit}(\pi)) \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The #1 (classical) operation of intersecting the set of even-sided figures with the set of solid figures to give the set of even-sided solid figures is represented as the conjunction:

$$\text{In}(\Delta E) \wedge \text{In}(\Delta S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The #2 (quantum) operation of ‘sharpening’ or ‘rendering more definite’ ‘the solid figure’ u_S to ‘the even-sided solid figure’ $u_{\{u_2, u_4\}}$, so $u_{\{u_2, u_4\}} \triangleleft u_S$ (suggested reading: $u_{\{u_2, u_4\}}$ is a projection of u_S) is represented as:

$$\text{In}(E \times E) \wedge \text{In}(S \times S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

But there is a better way to represent ‘sharpening’ using matrix multiplication instead of just the logical operation \wedge on matrices, and it foreshadows the measurement operation in QM. The matrix $\text{In}(\Delta E) = P_E$ is a projection matrix, i.e., the diagonal matrix with diagonal entries $\chi_E(u_i)$ so $P_E |S\rangle = |E \cap S\rangle$. Then the result of the projection-sharpening can be represented as:

$$\begin{aligned} |E \cap S\rangle (|E \cap S\rangle)^t &= P_E |S\rangle (P_E |S\rangle)^t = P_E |S\rangle (|S\rangle)^t P_E \\ &= P_E \text{In}(S \times S) P_E = \text{In}(E \times E) \wedge \text{In}(S \times S). \end{aligned}$$

Under the #2 interpretation, the parity-sharpening, parity-differentiation, or *parity-measurement* of ‘the solid figure’ by both parities is represented as:

$$\begin{aligned} \text{In}(\text{indit}(\pi)) \wedge \text{In}(S \times S) &= P_O \text{In}(S \times S) P_O + P_E \text{In}(S \times S) P_E \\ &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The results are ‘the even-sided solid figure’ $u_{\{u_2, u_4\}}$ and ‘the odd-sided solid figure’ $u_{\{u_3\}} = u_3$. The important thing to notice is the action on the off-diagonal elements where the action $1 \rightsquigarrow 0$ in the j, k -entry means that u_j and u_k have been deblobbed, decohered, distinguished, or differentiated—in this case by parity:

$$\begin{aligned} \text{In}(S \times S) &\rightsquigarrow \text{In}(\text{indit}(\pi)) \wedge \text{In}(S \times S) \\ &= P_O \text{In}(S \times S) P_O + P_E \text{In}(S \times S) P_E \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 \overset{\text{deblob}}{\rightsquigarrow} 0 & 1 \\ 0 & 1 \overset{\text{deblob}}{\rightsquigarrow} 0 & 1 & 1 \overset{\text{deblob}}{\rightsquigarrow} 0 \\ 0 & 1 & 1 \overset{\text{deblob}}{\rightsquigarrow} 0 & 1 \end{bmatrix}. \end{aligned}$$

We could also classify the figures as to having 4 or fewer sides ("few sides") or not ("many sides") so that partition is $\sigma = \{\{u_1, u_2\}, \{u_3, u_4\}\}$ which is represented by:

$$\text{In}(\text{indit}(\sigma)) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and}$$

$$\text{In}(\text{indit}(\sigma)) \wedge (\text{In}(\text{indit}(\pi)) \wedge \text{In}(S \times S)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{In}(\Delta S).$$

Thus parity and few-or-many-sides properties suffice to classify the solid figures uniquely and thus to yield the representation $\text{In}(\Delta S)$ of the distinct elements of $S = \{u_2, u_3, u_4\}$. Thus making all the distinctions (i.e., decohering the entities that cohered together in u_S) takes $\text{In}(S \times S) \rightsquigarrow \text{In}(\Delta S)$.

In QM jargon, the parity and few-or-many-sides attributes constitute a "complete set of commuting operators" (CSCO) so that measurement of 'the solid figure' by those observables will take 'the solid figure,' to the separate eigen-solid-figures: 'the few- and even-sided solid figure' (the square u_2), 'the many- and odd-sided solid figure' (the pentagon u_3), and 'the many- and even-sided solid figure' (the hexagon u_4).

10 From Incidence to Density Matrices

The incidence matrices $\text{In}(\Delta S)$ and $\text{In}(S \times S)$ can be turned into *density matrices* by dividing through by their trace:

$$\rho(\Delta S) = \frac{1}{\text{tr}[\text{In}(\Delta S)]} \text{In}(\Delta S) \text{ and } \rho(S) = \frac{1}{\text{tr}[\text{In}(S \times S)]} \text{In}(S \times S).$$

In terms of probabilities, this means treating the outcomes in S as being equiprobable with probability $\frac{1}{|S|}$. But now we have the #1 and #2 interpretations of the sample space for finite discrete probability theory.

1. The #1 (classical) interpretation, represented by $\rho(\Delta S)$, is the classical version with S as the sample space of outcomes. For instance, the 6×6 diagonal matrix with diagonal entries $\frac{1}{6}$ is "the statistical mixture describing the state of a classical dice [die] before the outcome of the throw" [1, p. 176];
2. The #2 (quantum) interpretation replaces the "sample space" with the one indefinite 'the sample outcome' u_S represented by $\rho(S)$ (like 'the outcome of throwing a die') and, in a trial, the indefinite outcome u_S 'sharpen to' or becomes a definite outcome $u_i \in S$ with probability $\frac{1}{|S|}$.

Let $f : U \rightarrow \mathbb{R}$ be a real-valued random variable with distinct values ϕ_i for $i = 1, \dots, m$ and let $\pi = \{B_i\}_{i=1, \dots, m}$ where $B_i = f^{-1}(\phi_i)$, be the partition of U according to the values. The classification of $\rho(S)$ according to the different values is: $\text{In}(\text{indit}(\pi)) \wedge \rho(S)$ which distinguishes the elements of S that have different f -values. If P_{B_i} is the diagonal (projection) matrix with diagonal elements $(P_{B_i})_{jj} = \chi_{B_i}(u_j)$, then the probability of a trial returning a u_j with $f(u_j) = \phi_i$ is:

$$\Pr(\phi_i|S) = \text{tr}[P_{B_i}\rho(S)].$$

For instance, in the previous example, where $f : U \rightarrow \mathbb{R}$ gives the parity partition π with the two values ϕ_{odd} and ϕ_{even} , then:

$$P_{\text{even}}\rho(S) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

so $\text{tr}[P_{\text{even}}\rho(S)] = \frac{2}{3}$ which is the conditional probability of getting ‘the even-sided solid figure’ starting with ‘the solid figure’ in the #2 (quantum) interpretation. And under the #1 (standard) interpretation, $\Pr(\phi_{\text{even}}|S) = \text{tr}[P_{\text{even}}\rho(\Delta S)] = \frac{2}{3}$ which is the probability of getting *an* even-sided solid figure starting with the set of solid figures.

These two interpretations of finite discrete probability theory extend easily to the case of point probabilities p_j for $u_j \in U$, where:

1. $(\rho(\Delta S))_{jj} = \chi_S(u_j) p_j / \Pr(S)$, so $\text{tr}[P_{\text{even}}\rho(\Delta S)] =$ probability of getting *an* even-sided solid figure starting with the set of solid figures, and
2. $(\rho(S))_{j,k} = \chi_S(u_j) \chi_S(u_k) \sqrt{p_j p_k} / \Pr(S)$, so $\text{tr}[P_{\text{even}}\rho(S)] =$ probability of getting ‘the even-sided solid figure’ starting with ‘the solid figure.’

The whole of finite discrete probability theory can be developed in this manner, *mutatis mutandis*, for the #2 interpretation paradigms.

11 Density matrices in Quantum Mechanics

The jump to quantum mechanics (QM) is to replace the binary digits like 0, 1 in incidence matrices or reals $\sqrt{p_j p_k}$ in ‘classical’ density matrices by complex numbers. Instead of the set S represented by a column $|S\rangle$ of 0, 1, we have a normalized column $|\psi\rangle$ of complex numbers α_j whose absolute squares are probabilities: $|\alpha_j|^2 = p_j$, e.g.,

$$|S\rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow |\psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

where $\alpha_1 = 0$ and $|\alpha_j|^2 = p_j$ for $j = 2, 3, 4$.

1. The *density matrix* $\rho(\Delta\psi)$ has the absolute squares $|\alpha_j|^2 = p_j$ laid out along the diagonal.
2. The *density matrix* $\rho(\psi)$ has the j, k -entry as the product of α_j and α_k^* (complex conjugate of α_k), where $p_j = \alpha_j^* \alpha_j = |\alpha_j|^2$.

Thus:

$$\rho(\Delta\psi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \text{ and } \rho(\psi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_2 & \alpha_2 \alpha_3^* & \alpha_2 \alpha_4^* \\ 0 & \alpha_3 \alpha_2^* & p_3 & \alpha_3 \alpha_4^* \\ 0 & \alpha_4 \alpha_2^* & \alpha_4 \alpha_3^* & p_4 \end{bmatrix}.$$

[The] off-diagonal terms of a density matrix...are often called *quantum coherences* because they are responsible for the interference effects typical of quantum mechanics that are absent in classical dynamics. [1, p. 177]

The classifying or measuring operation $\text{In}(\text{indit}(\pi)) \wedge \rho(\psi)$ could still be defined taking the minimum of corresponding entries in absolute value, but in QM it is defined as the *Lüders mixture operation* [1, p. 279]. If $\pi = \{B_1, \dots, B_m\}$ is a partition according to the eigenvalues ϕ_1, \dots, ϕ_m on $U = \{u_1, \dots, u_n\}$ (where U is an orthonormal basis set for the observable being measured), let P_{B_i} be the diagonal (projection) matrix with diagonal entries $(P_{B_i})_{jj} = \chi_{B_i}(u_j)$. Then $\text{In}(\text{indit}(\pi)) \wedge \rho(\psi)$ is obtained as:

$$\sum_{B_i \in \pi} P_{B_i} \rho(\psi) P_{B_i}$$

The Lüders mixture.

The probability of getting the result ϕ_i is:

$$\Pr(\phi_i|\psi) = \text{tr}[P_{B_i}\rho(\psi)].$$

12 A Pop Science Interlude

The popular science version of the simplest case is Schrödinger's cat.



Figure 7: Usual "And" version of Schrödinger's cat.

This version of Schrödinger's cat as being "Dead & Alive" is like the usual mis-interpretation of the unobserved particle as going through "Slit 1 & Slit 2" in the double slit experiment. But the cat is not definitely alive **and** definitely dead at the same time. The quantum version is that the cat is indefinite between those two definite possibilities; it's in cat-limbo.

$$\text{Schrödinger's cat} = \text{dead-cat} \boxplus \text{live-cat}.$$

It would be more accurate to say "Dead or Alive—but neither definitely," a "disjunctive fact" [12].



Figure 8: The disjunctive cat.

Technically the state vector is:

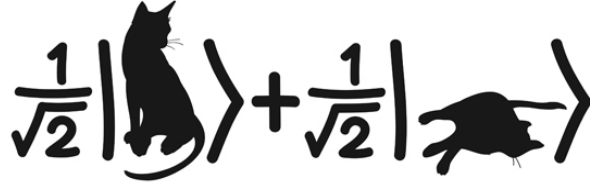


Figure 9: Schrödinger's cat state vector.

Using density matrices, we would represent Schrödinger's cat as being in the state:

$$\rho(cat) = \begin{bmatrix} \frac{1}{2}live & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}dead \end{bmatrix}.$$

13 Simplest Quantum Example

Consider a system with two spin-observable σ eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ (like electron spin up or down along the z -axis) where the given normalized superposition state is $|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle = \begin{bmatrix} \alpha_{\uparrow} \\ \alpha_{\downarrow} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ so the density matrix is $\rho(\psi) = \begin{bmatrix} p_{\uparrow} & \alpha_{\uparrow}\alpha_{\downarrow}^* \\ \alpha_{\downarrow}\alpha_{\uparrow}^* & p_{\downarrow} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ where $p_{\uparrow} = \alpha_{\uparrow}\alpha_{\uparrow}^*$ and $p_{\downarrow} = \alpha_{\downarrow}\alpha_{\downarrow}^*$. The measurement in that spin-observable σ goes from $\rho(\psi)$ to

$$\text{In (indit } (\sigma)) \wedge \rho(\psi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} p_{\uparrow} & \alpha_{\uparrow}\alpha_{\downarrow}^* \\ \alpha_{\downarrow}\alpha_{\uparrow}^* & p_{\downarrow} \end{bmatrix} = \begin{bmatrix} p_{\uparrow} & 0 \\ 0 & p_{\downarrow} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \rho(\Delta\psi).$$

Or using the Lüders mixture operation:

$$\begin{aligned} & P_{\uparrow}\rho(\psi)P_{\uparrow} + P_{\downarrow}\rho(\psi)P_{\downarrow} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{\uparrow} & \alpha_{\uparrow}\alpha_{\downarrow}^* \\ \alpha_{\downarrow}\alpha_{\uparrow}^* & p_{\downarrow} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{\uparrow} & \alpha_{\uparrow}\alpha_{\downarrow}^* \\ \alpha_{\downarrow}\alpha_{\uparrow}^* & p_{\downarrow} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} p_{\uparrow} & 0 \\ 0 & p_{\downarrow} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \rho(\Delta\psi). \end{aligned}$$

The two versions of $S = U$ give us two versions of finite discrete probability theory where: #1) U is the sample space or #2) u_U is the sample outcome.

1. The #1 classical version is the usual version which in this case is like flipping a fair coin and getting head or tails with equal probability.

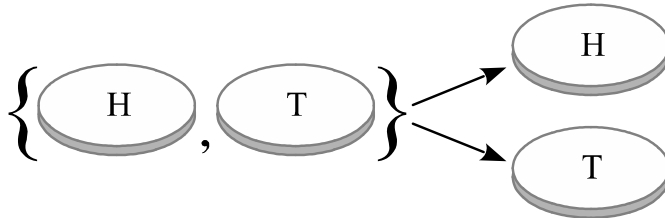


Figure 10: Outcome set for classical coin-flipping trial.

2. The #2 quantum version starts with the indefinite entity $u_U = \boxplus \{u_i \in U\}$, ‘the (indefinite) outcome’, and a trial renders it into one of the definite outcomes u_i with some probability p_i so that u_U could be represented by the density matrix $\rho(U)$ where $(\rho(U))_{jk} = \sqrt{p_j p_k}$. In this case, this is like a coin $u_{\{H,T\}}$ with the difference between heads or tails rendered indefinite or blobbed out, and the trial results in it sharpening to definitely heads or definitely tails with equal probability.

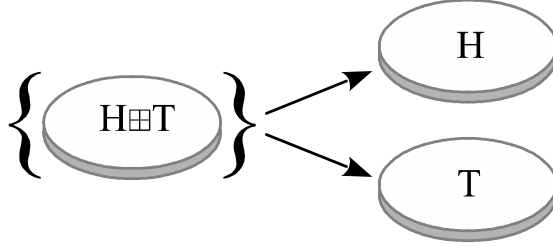


Figure 11: ‘the outcome state’ for quantum coin-flipping trial.

Experimentally, it is not possible to distinguish between the #1 and #2 versions *by σ -measurements*. But in QM the two states $\rho(\Delta\psi)$ and $\rho(\psi)$ *can* be distinguished by measuring other observables like spin along a different axis [1, p. 176]. Thus we *know* in QM which version is the superposition (pure) state $|\psi\rangle = \begin{bmatrix} \alpha_{\uparrow} \\ \alpha_{\downarrow} \end{bmatrix}$; it is *the #2 blob-state* $\rho(\psi)$.

14 Conclusions

Quantum mechanics texts usually mention several interpretations such as the Copenhagen, many-worlds, or hidden-variables interpretations. Now that we have established a bridge from abstraction in mathematics to indefinite states in QM, we may (for fun) cross the bridge in the opposite direction. For instance, in the many-worlds (or many-minds) interpretation, $1 \in \pi_0(A) \cong \mathbb{Z}$ would refer to a different specific coordinatized "once clockwise around the hole" path in each different world (or mind).

Shimony, however, suggests the *Literal or Objective Indefiniteness Interpretation*—which we have seen is suggested by the mathematics itself.

But the mathematical formalism ... suggests a philosophical interpretation of quantum mechanics which I shall call "the Literal Interpretation." ...This is the interpretation resulting from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete. [11, pp. 6-7]

We have approached QM by starting with the logical situation of a universe U of distinct entities. Given a property $S(x)$ on U , we can associate with it:

1. the #1 abstract object $S = \{u_i \in U | S(u_i)\}$, the set of $S(x)$ -entities, or
2. the #2 abstract object $u_S = \boxplus \{u_i \in U | S(u_i)\}$ which is the abstract entity expressing the properties common to the $S(x)$ -entities but "abstracting away from," "rendering indefinite," "cohering together," or "blobbing out" the differences between those entities.

We argued that the mathematical formalisms of incidence matrices and then density matrices can be used to formalize the two representations:

1. #1 representation as $\text{In}(\Delta S)$ or $\rho(\Delta\psi)$; and
2. #2 representation as $\text{In}(S \times S)$ or $\rho(\psi)$.

This dove-tailed precisely into usual density-matrix treatment in QM of quantum states $|\psi\rangle$ as $\rho(\psi)$ which, as suggested by Shimony, can be interpreted as *objectively* indefinite states.

Yet since the ancient Greeks, we have the #2 Platonic notion of the abstract paradigm-universal ‘the S -entity’, definite on what is common to the members of a set S and indefinite on where they differ, so the connection that may help to better understand quantum mechanics is:

The paradigm u_S , ‘the S -entity’ represented by $\text{In}(S \times S) \iff$ the superposition state ψ
represented by the density matrix $\rho(\psi)$.

This recalls Whitehead’s quip that Western philosophy is "a series of footnotes to Plato." [14, p. 39]

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