The Joy of Hets:
Heteromorphisms in Category Theory

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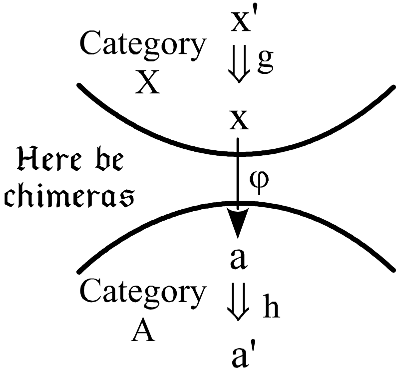
Heteromorphisms: I

- In mathematical practice, a *heteromorphism* or *het* is an object-to-object morphism between objects in **different** categories, e.g., a set-to-group map.
- The composition of homomorphisms (homs) is mathematically described by a *hom-bifunctor* $\mathrm{Hom}_X : X^{op} \times X \to Set$.
- The composition of heteromorphisms is mathematically described by a *het-bifunctor* $\mathrm{Het}_{X \to A} : X^{op} \times A \to Set$.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Heteromorphisms: III

- In the textbook definitions of category theory (CT), heteromorphisms are not officially recognized; the only object-to-object morphisms are the homo-morphisms between objects of the same category.
- Het-bifunctors = *Set-valued profunctors* (e.g., the Australian School) = *distributors* (e.g., Bénabou) = Lawvere's *bimodules*.
- Homs are part of math practice, not just elements of $\mathrm{Hom}_X(x, x')$.
- Hets are also part of math practice, but homs-only CT sees them only as elements of some profunctor $W(x, a)$.

- By the Yoneda-Grothendieck Lemma, any value of a (contravariant) functor to *Set* like $\text{Het}(-, a) : X^{op} \longrightarrow Set$ is isomorphic to the set of natural transformations $n.t.\{\text{Hom}_X(-, x), \text{Het}(-, a)\}$ but no one thinks that, say, a set-to-group het $x \longrightarrow a$ "is" a natural transformation $\text{Hom}_X(-, x) \longrightarrow \text{Het}(-, a)$.

Hets give natural treatment of adjoints: I

- Given a bifunctor Het : $X^{op} \times A \rightarrow Set$, it is *left representable* if for each $x \in X$, $\text{Het}(x, \_) : A \rightarrow Set$ is representable, i.e., $\exists F(x) \in A$ with a natural isomorphism:

$$\text{Hom}_A(F(x), \_) \cong \text{Het}(x, \_).$$

- Universal mapping problem formulation of left-representation: for each $x \in X$, there exists an object $F(x) \in A$ and a canonical het $\eta_x : x \rightarrow F(x)$ such that given any het $\varphi : x \rightarrow a$ (to any $a \in A$), there exists a unique hom $\underline{f} : F(x) \Longrightarrow a$ such that the following diagram commutes (single arrows $\rightarrow$ and Greek letters for hets; thick arrows $\Rightarrow$ and Latinic letters for homs):

Hets give natural treatment of adjoints: II

$$\begin{tikzcd} x \arrow[d, "\eta_x"'] \arrow[dr, "\varphi"] & \\ F(x) \arrow[r, Rightarrow, "\exists! f"'] & a \end{tikzcd}$$

Left-representation as solution to UMP.

- Given a bifunctor Het : $X^{op} \times A \to Set$, it is *right representable* if for each $a \in A$, Het $(\_, a) : X^{op} \to Set$ is representable, i.e., $\exists G(a) \in X$ with a natural isomorphism:

$$\text{Het}(\_, a) \cong \text{Hom}_X(\_, G(a)).$$

Hets give natural treatment of adjoints: III

- UMP form of right-representation: for each $a \in A$, there exists an object $G(a) \in X$ and a canonical het $\varepsilon_a : G(a) \to a$ such that for any het $\varphi : x \to a$ (from any $x \in X$), there exists a unique hom $\bar{f} : x \Longrightarrow G(a)$ such that the following diagram commutes:

$$\begin{array}{ccc} x & \overset{\exists!\bar{f}}{\Longrightarrow} & G(a) \\ & \searrow & \downarrow \varepsilon_a \\ & \varphi & \\ & & a \end{array}$$

Right representation as solution to Co-UMP.

- If a bifunctor Het : $X^{op} \times A \to Set$ is both left and right representable, then the two functors $F : X \to A$ and $G : A \to X$ giving the representing objects are adjoints.

Hets give natural treatment of adjoints: IV

$$\mathrm{Hom}_A(F(x), a) \cong \mathrm{Het}(x,a) \cong \mathrm{Hom}_X(x, G(a)).$$

- This het characterization of adjoints was first given by Bodo Pareigis in his 1970 text, *Categories and Functors* where hets are just called "morphisms" (rediscovered and developed in: Ellerman, David. 2006. “A Theory of Adjoint Functors” In *What Is Category Theory?*, G. Sica ed., 127–83. Milan: Polimetrica). I am unaware of any other text giving this characterization (since it requires explicit recognition of "morphisms" $x \to a$).
- Splicing the two representation diagrams together at the common diagonal gives the simplest diagram for an adjunction:

Hets give natural treatment of adjoints: V

$$\begin{array}{ccc} x & \overset{\exists!\,\bar{f}}{\Longrightarrow} & G(a) \\ \eta_x \downarrow & \searrow^{\varphi} & \downarrow^{\varepsilon_a} \\ F(x) & \underset{\exists!\,\underline{f}}{\Longrightarrow} & a \end{array}$$

Adjunctive square diagram.

Aspects of het treatment of adjoints: I

- Separating "wheat from chaff":
  - Using hets, an adjunction can be factored into two "semi-adjunctions" giving the UMPs for each functor:

$$\mathrm{Hom}_A(F(x), a) \cong \mathrm{Het}(x,a) \text{ and } \mathrm{Het}(x,a) \cong \mathrm{Hom}_X(x, G(a)).$$

$$\begin{tikzcd}
x \arrow[d, "\eta_x"'] \arrow[dr, "\varphi"] & \\
F(x) \arrow[r, Rightarrow, "\exists! f"'] & a
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, Rightarrow, "\exists! \bar{f}"] \arrow[dr, "\varphi"'] & G(a) \arrow[d, "\varepsilon_a"] \\
 & a
\end{tikzcd}$$

Aspects of het treatment of adjoints: II

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  - Often only one adjoint gives the real "meat" (or "wheat") of the adjunction, while the other functor is more like a trivial auxiliary device ("chaff"), e.g., forgetful or diagonal functor, to fill out the adjunction.
  - Ordinary homs-only CT cannot even formulate the important part of the adjunction *by itself*.
  - Since the het treatment consists of two independently-stated representations, it can state the important part by itself as a left or right representation.
  - Experiment: try to find an ordinary algebra text (not specifically on CT) that states the UMP for the free group by using the underlying-set functor.

Aspects of het treatment of adjoints: III

- They all (?) state the left representation property: given a set x, there is a group $F(x)$ and a canonical mapping $x \to F(x)$ such that for any other mapping $x \to g$ from x to any group g, there is a unique group homomorphism $F(x) \Rightarrow g$ such that the following triangle commutes.

$$\begin{tikzcd} x \arrow[d] \arrow[dr] & \\ F(x) \arrow[r, Rightarrow, "\exists!"'] & g \end{tikzcd}$$

- - That is the way the "working mathematician" states the free-group UMP and that is the left representation using hets.

Aspects of het treatment of adjoints: IV

- Comparison of diagrams: Although we may be accustomed to it, the usual adjunction diagram for, say, the UMP of the free-group functor is the rather complicated over-and-back diagram–in contrast to the left-rep diagram:

$$
\begin{tikzcd}
x \arrow[r, Rightarrow, "\eta_x"] \arrow[d, equal] & G(F(x)) \arrow[d, Rightarrow, "G(\underline{f})"'] & F(x) \arrow[d, Rightarrow, "\exists!\underline{f}"'] \\
x \arrow[r, Rightarrow, "f"] & G(a) & a
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[d, "\eta_x"'] \arrow[dr, "\varphi"] & \\
F(x) \arrow[r, Rightarrow, "\exists!\underline{f}"'] & a
\end{tikzcd}
$$

UMP of left adjoint: Over-and-back diagram vs. left-representation diagram.

- Directionality of an adjunction $\mathrm{Hom}_A(F(x), a) \cong \mathrm{Hom}_X(x, G(a))$:

Aspects of het treatment of adjoints: V

- Experiment: Ask your students or colleagues without a black-belt in CT: Is an adjunction a symmetrical situation between the categories or is that a directionality from one category to the other?
- Answer: Look at the adjunctive square diagram; all three hets go from X to A.

$$\begin{array}{ccc} x & \overset{\exists!\,\bar{f}}{\Longrightarrow} & G(a) \\ \eta_x \downarrow & \searrow^{\varphi} & \downarrow \varepsilon_a \\ F(x) & \underset{\exists!\,\underline{f}}{\Longrightarrow} & a \end{array}$$

Adjunctive square diagram.

Het-avoidance devices: Forgetful functors: I

- One common type of UMP is a left representation of hets like "injection of generators" (going from a category of less-structured objects to a category of more-structured objects).

Forgetful functor	*Left adjoint F*	*Unit of adjunction*
$U : R\text{-}\mathbf{Mod} \to \mathbf{Set}$	$X \mapsto FX$ Free R-module, basis X	$j : X \to UFX$ (cf. § III.1) "insertion of generators"
$U : \mathbf{Cat} \to \mathbf{Grph}$	$G \mapsto CG$ Free category on graph G	$G \to UCG$ "insertion of generators"
$U : \mathbf{Grp} \to \mathbf{Set}$	$X \mapsto FX$ Free group, generators $x \in X$	$X \to UFX$ "insertion of generators"
$U : \mathbf{Ab} \to \mathbf{Set}$	$X \mapsto F_a X$ Free abelian group on X	"insertion of generators"

Het-avoidance devices: Forgetful functors: II

- The hets can be avoided using a forgetful functors with trivial right representations to form a homs-only adjunction.
- Experiment: try to find an ordinary algebra text giving the UMP of canonically going from the less- to the more-structured object that uses the forgetful functor.

Het-avoidance devices: Diagonal functors: I

- Another common form of hets are cones and cocones used in the right or left representations giving the UMPs for limits and colimits (respectively).
- The hets can be avoided using diagonal functors with trivial left or right representations to form homs-only adjunctions.

Diagonal functor	*Adjoint*	*Unit*	*Counit*
$\Delta : C \to C \times C$	Left: Coproduct	(pair of) injections	"folding" map
	$\amalg : C \times C \to C$	$i : a \to a \amalg b$	$c \amalg c \to c$
	$\langle a, b \rangle \mapsto a \amalg b$	$j : b \to a \amalg b$	$ix \mapsto x,\ jx \mapsto x$
	Right: Product	Diagonal arrow	(pair of) projections
	$\Pi : C \times C \to C$	$\delta_c : c \to c \times c$	$p : a \times b \mapsto a$
	$\langle a, b \rangle \mapsto a \times b$	$x \mapsto \langle x, x \rangle$	$q : a \times b \mapsto b$

Het-avoidance devices: Diagonal functors: II

- Experiment: try to find an ordinary algebra text giving the UMP of say products or coproducts that uses the diagonal functor.
- Example: In Mac Lane's 1948 paper *Groups, Categories, and Duality* giving the UMP for products (a decade before Kan's 1958 paper on adjoint functors), Mac Lane uses cones as hets which he called "systems" of maps and gives the UMP sans diagonal functors.

Hets as "homs" in a collage category

- Pareigis and many later category theorists have pointed out that hets Het : $X^{op} \times A \longrightarrow Set$ could always be presented as homs in a larger cograph or collage category $X \bigstar^{\text{Het}} A$ whose objects are the disjoint union $X \biguplus A$ and whose homs come in three different types:
  - for $x, x' \in X$, $\text{Hom}_{X \bigstar^{\text{Het}} A}(x, x') = \text{Hom}_X(x, x')$;
  - for $x \in X$ and $a \in A$, $\text{Hom}_{X \bigstar^{\text{Het}} A}(x, a) = \text{Het}(x, a)$; and
  - for $a, a' \in A$, $\text{Hom}_{X \bigstar^{\text{Het}} A}(a, a') = \text{Hom}_A(a, a')$.
- This device completely violates spirit of one motivation for CT, the Erlangen Program.
- None of the UMPs using hets can be reformulated using "homs" in general from $X \bigstar^{\text{Het}} A$ without stating which of the three types of homs are used so it is largely a verbal circumlocution.

Tensor products–where het-avoidance doesn't work: I

- Tensor product = most important example of a UMP not part of an adjunction–so hets are unavoidable.
- Hets = bilinear maps $\varphi : \langle A, B \rangle \to C$ for R-modules A, B, C.
- The tensor product functor $\otimes : Mod_R \times Mod_R \to Mod_R$ given by $\langle A, B \rangle \longmapsto A \otimes B$ gives a left representation:

$$\mathrm{Hom}_{Mod_R}(A \otimes B, C) \cong \mathrm{Het}(\langle A, B \rangle, C).$$

$$\begin{tikzcd} \langle A,B \rangle \arrow[d, "\eta_{\langle A,B \rangle}"'] \arrow[dr, "\varphi"] & \\ A \otimes B \arrow[r, Rightarrow, "\exists! f"'] & C \end{tikzcd} \qquad \begin{tikzcd} \{*\} \arrow[r, Rightarrow, "\ulcorner \eta_{\langle A,B \rangle} \urcorner"] \arrow[d, equal] & \mathrm{Het}(\langle A,B \rangle, A \otimes B) \arrow[d, Rightarrow, "\mathrm{Het}(\langle A,B \rangle, f)"'] \\ \{*\} \arrow[r, Rightarrow, "\ulcorner \varphi \urcorner"] & \mathrm{Het}(\langle A,B \rangle, C) \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[d, Rightarrow, "\exists! f"'] \\ C \end{tikzcd}$$

Left rep-diagram and Mac Lane's homs-only version.

Tensor products–where het-avoidance doesn't work: II

- Mac Lane and Birkhoff's *Algebra* textbook uses hets starting with the case of a K-module A (K a comm. ring) where $A \otimes K \cong A$: for any K-module C, "the arbitrary bilinear function h is expressed as a composite $h = t \circ h_0$ with the fixed bilinear function h_0, as in the commutative diagram"

$$\begin{array}{ccc} A \times K & & \\ {}^{h_0}\downarrow & \searrow^{h} & \\ A \otimes K & \underset{\exists ! t}{\Rightarrow} & C \end{array}.$$

Pointed sets as right-rep for partial functions: I

- Let Het : $Set^{op} \times Set \to Set$ be defined by: Het(x, y) = partial functions $x \to y$.
- Let $(^*) : Set \to Set$ be the *pointing functor* that adds a "garbage point" to a set so $y^* = y \uplus \{*_y\}$.

$$\mathrm{Het}(x,y) \cong \mathrm{Hom}_{Set}(x, y^*)$$

$$\begin{array}{ccc} x & \overset{\exists! f}{\Longrightarrow} & y^* \\ & \searrow & \downarrow \varepsilon_y \\ & \varphi & \\ & & y \end{array}$$

Pointing functor gives right rep for partial functions as hets.

Pointed sets as right-rep for partial functions: II

- Special case: $1^* \cong 2$ where $\mathrm{Het}(x,1) \cong \wp(x)$ (contravariant power set functor) so right rep associates characteristic functions with subsets.

$$\begin{array}{ccc} x & \overset{\exists! f}{\Longrightarrow} & 1^* = 2 \\ & \searrow_{\varphi} & \downarrow \varepsilon_y \\ & & 1 \end{array}$$

$$\wp(x) \cong \mathrm{Het}(x,1) \cong \mathrm{Hom}_{Set}(x,1^*) \cong \mathrm{Hom}_{Set}(x,2).$$

- This shows that Lawvere's subset classifier isn't the only way to show that the powerset functor is representable.

Anglo-American versus French CT: I

- Aside from being co-founder of category theory, Mac Lane has set the standard in Anglo-American CT as the homs-only CT.
- The homs-only notion of adjoints is seen as the most fundamental contribution of CT:

Anglo-American versus French CT: II

Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. [Steve Awodey (Mac Lane's last student)]

To some, including this writer, adjunction is the most important concept in category theory. [Richard J. Wood]

The isolation and explication of the notion of adjointness is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas. [Robert Goldblatt]

Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics. [Paul Taylor]

Anglo-American versus French CT: III

- There is an old controversy, largely personalized as between Mac Lane and Bourbaki, as to why Bourbaki didn't use CT.
- That is **not** the controversy discussed here.
- There is a more subtle controversy within category theory where the person on the other side is Grothendieck.

Anglo-American versus French CT: IV

As we can see by looking at his [Grothendieck's] lectures in the Séminaire Bourbaki from 1957 until 1962, the notion of representable functors became one of the main tools he used.... It is far from clear why Grothendieck decided to use this notion instead of, say, adjoint functors,.... It is also clear from the various seminars that Grothendieck thought in terms of universal "problems", that is he tried to formulate the problems he was working on in terms of a universal morphism: finding a solution to the given problem amounted to finding a universal morphism in the situation. Grothendieck saw that the latter notion was subsumed under the notion of representable functor. [Jean-Pierre Marquis]

Anglo-American versus French CT: V

- Grothendieck did not take adjoints as the fundamental concept; instead it was the first concept treated in EGA.

CHAPITRE 0

Préliminaires

§ 1. Foncteurs représentables

1.1. Foncteurs représentables

- Grothendieck (as the representative of the French school of CT) took representable functors (and universal mapping problems) as the fundamental contribution of CT with adjoints appearing as the special case of a particularly nice bi-representation.

Anglo-American versus French CT: VI

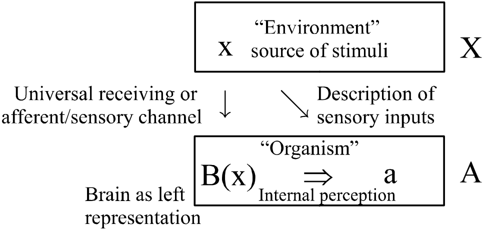
- The French treatment of UMPs, starting with Pierre Samuel's 1948 paper "On Universal Mappings and Free Topological Groups," routinely used hets and distinguished them from homs by using Greek and Latinic letters respectively.
- Hence our heterodox journey reveals a difference between:
    - The Anglo-American "Orthodox" school represented by Mac Lane which emphasizes adjoints and eschews hets;
    - The French school represented by Grothendieck which emphasizes representable functors and universal mapping problems, and which routinely uses hets.

Anglo-American versus French CT: VII

- Moreover, in the "foundational debates" between category theory and set theory, it was always emphasized that CT, unlike set theory, reflects the mathematical practice of the "working mathematician."
- As we have noted, the working mathematician routinely uses hets.

Appendix: Perception as a left representation

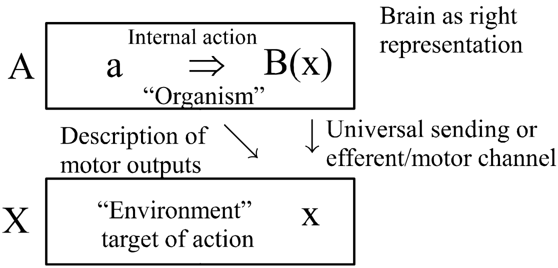
- One application is hets-treatment of adjoints. Second application is cognate notion of a *brain functor*.
- Using a left representation to model the intentionality of perception (seeing is "seeing-as").



- Het $x \rightarrow a$ represents uninterpreted sensory input; factorization through $B(x) \Rightarrow a$ represents the recognition, understanding, or interpretation of the sensory input.

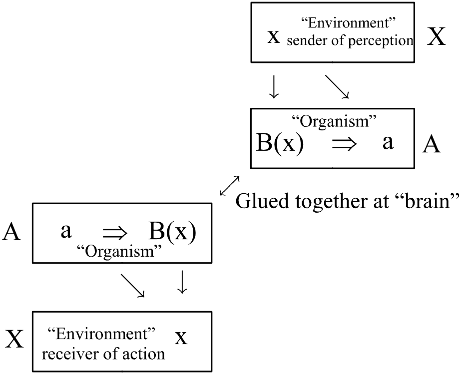
Action as right representation

- Dually, a right representation models the intentionality of action.



- The het $a \rightarrow x$ represents the motor behavior; factorization through $a \Rightarrow B(x)$ represents the intentionality of action (as opposed, say, to some reflex behavior).

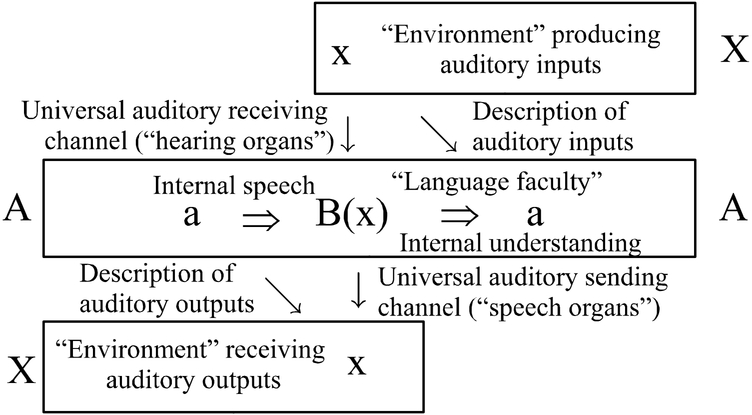
- Adjunction = two functors representing one-way hets.
- Brain functor = one functor representing both-way hets.



Brain functor: II

- Adjunction diagram = adjunctive square (gluing together at common het $x \to a$).
- Brain functor diagram = butterfly diagram (gluing together at common representing object $B(x)$).

Brain functor: III



Butterfly diagram illustrating brain as language faculty

Brain functor: IV

- Wilhelm von Humboldt recognized the symmetry between the speaker and listener, which in the same person is abstractly represented as the dual functions of the "selfsame power" of the language faculty in the above butterfly diagram.

Nothing can be present in the mind (Seele) that has not originated from one's own activity. Moreover understanding and speaking are but different effects of the selfsame power of speech. Speaking is never comparable to the transmission of mere matter (Stoff). In the person comprehending as well as in the speaker, the subject matter must be developed by the individual's own innate power. What the listener receives is merely the harmonious vocal stimulus. [The Nature and Conformation of Language, 1836]

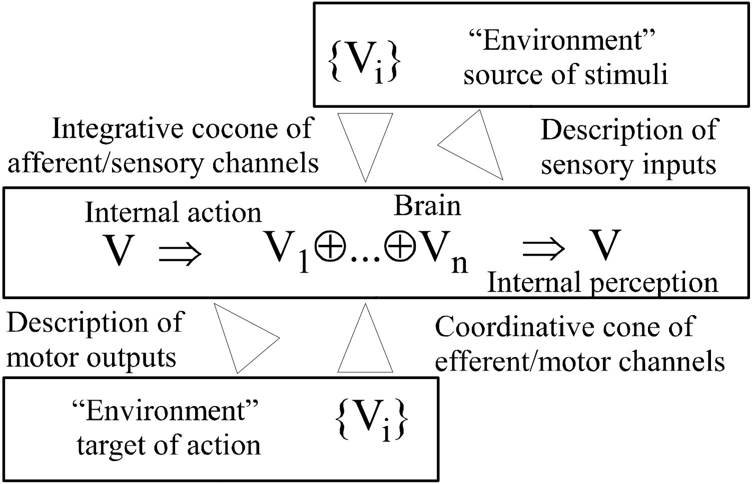
Brain functor: V

- Brain functor models difference between mere sensory stimulus and intentionality of understanding.
- Brain functor models difference between mere behavior and intentionality of action.
- Brain functors uses CT duality to model the old "duality" between afferent/sensory systems and efferent/motor systems.

Mathematical example of brain functor: I

- Any functor with both a left and right adjoint is a brain functor, but many are trivial examples, e.g., diagonal functors.
- Best non-trivial example seems to be the *biproduct* $V_1 \oplus \ldots \oplus V_n \cong \prod_i V_i \cong \sum_i V_i$ of a finite set $\{V_i\}$ of vector spaces.

Mathematical example of brain functor: II



Mathematical example of brain functor: III

- The afferent function has to integrate many sensory inputs into a "perception."
- The efferent function has to coordinate many motor outputs into an "action."
- These integrative and coordinative functions are exactly represented by cocones and cones.