The Logic of Partitions

Introduction to the Dual of "Propositional" Logic

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Why Partition Logic took so long to develop

- Boolean logic mis-specified as logic of "*propositions*."
- Boolean logic correctly specified as logic of *subsets*.
- Valid formula $=_{df}$ formula that always evaluates to universe set U regardless of subsets of U substituted for variables.
- *Truth table* validity should be theorem, not definition, i.e., theorem that for validity it suffices to take $U = 1$, or to only substitute in U and $\emptyset$.
- Almost all logic texts *define* "tautology" as truth-table tautology.

One consequence: Renyi's Theorem took a century

- Boole developed (1850s) Boolean logic as logic of subsets, and then developed logical finite probability theory as normalized counting measure on subsets (events).
- As the mis-specification as propositional logic later dominated, it took a century (1961) to realize that the theorem (it suffices to substitute U and $\emptyset$) extends to valid statements in probability theory.

A GENERAL METHOD FOR PROVING THEOREMS IN PROBABILITY THEORY AND SOME APPLICATION

A. RÉNYI

Subsets category-theoretic dual to partitions

- Subsets have a CT-dual; propositions don't.
- CT duality gives subset-partition duality:
  - Set-monomorphism or injection determines a subset of its codomain (image);
  - Set-epimorphism or surjection determines a partition of its domain (inverse-image or coimage).
- In category theory, subsets generalize to subobjects or "parts".

"The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition." (Lawvere)

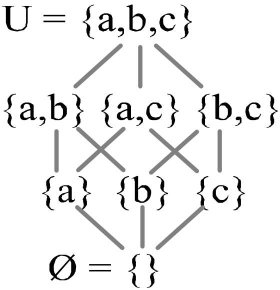
Duality:

Elements of a subset dual to *distinctions* of a partition

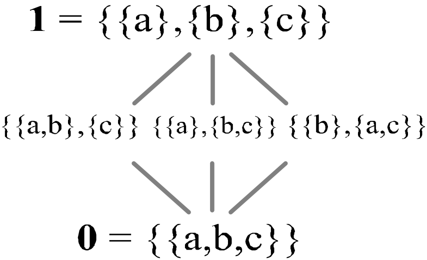
- A *partition* $\pi = \{B\}$ on a set U is a mutually exclusive and jointly exhaustive set of subsets or blocks B of U, a.k.a., an equivalence relation on U or quotient set of U.
- A *distinction* or *dit* of π is an ordered pair (u, u') with u and u' in distinct blocks of π.

	Subsets S of U	Partitions π on U
"Atoms"	Elements $u \in S$	Distinctions (u, u') of π
All atoms	All elements: U	All dits: discrete partition 1
No atoms	No elements: $\emptyset$	No dits: indiscrete partition 0
Partial order	Inclusion of elements	Inclusion of distinctions
Lattice	Boolean lattice	Partition lattice

The two lattices



Boolean lattice
of subsets of U



Partition lattice
of partitions on U

Algebras of Subsets and Partitions: I

- Given universe set U, there is the *Boolean algebra of subsets* $\wp(U)$ with inclusion as partial ordering and the usual union and intersection, and enriched with implication: $A \Longrightarrow B = A^c \cup B$.
- Given universe set U, there is the *algebra of partitions* $\Pi(U)$ with join and meet enriched by implication where refinement is the partial ordering.
  - Given partitions $\pi = \{B\}$ and $\sigma = \{C\}$, σ is *refined* by π, $\sigma \preceq \pi$, if for every block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$.
  - *Join* $\pi \vee \sigma$ is partition whose blocks are non-empty intersections $B \cap C$.

Algebras of Subsets and Partitions: II

- *Meet* $\pi \wedge \sigma$: define undirected graph on U with link between u and u' if they are in same block of π or σ. Then connected components of graph are blocks of meet.
- *Implication* $\sigma \implies \pi$ is the partition that is like π except that any block $B \in \pi$ contained in some block $C \in \sigma$ is discretized. Discretized B like a mini-**1** & Undiscretized B like a mini-**0** so $\sigma \Rightarrow \pi$ is an indicator function for (partial) refinement. Then

$$\sigma \preceq \pi \text{ iff } \sigma \Rightarrow \pi = \mathbf{1}.$$

- *Top* $\mathbf{1} = \{\{u\} : u \in U\}$ = discrete partition;
- *Bottom* $\mathbf{0} = \{U\}$ = indiscrete partition = "blob"

Tautologies in subset and partition logics: I

- A *subset tautology* is any formula which evaluates to U ($|U| \geq 1$) regardless of which subsets were assigned to the atomic variables.
- A *partition tautology* is any formula which always evaluates to $\mathbf{1}$ (the discrete partition) regardless of which partitions on U ($|U| \geq 2$) were assigned to the atomic variables.
- A *weak partition tautology* is a formula that is never indiscrete, i.e., never evaluates to indiscrete partition $\mathbf{0}$.
- For subset tautologies, it suffices to take $U = 1 = \{*\}$ so $\wp(1) = \{\emptyset, 1\}$ as in the truth tables with values 0 and 1.
- For $U = 2 = \{0, 1\}$ (any two element set), $\Pi(2) = \{\mathbf{0}, \mathbf{1}\}$ (indiscrete and discrete partitions) and partition ops are Boolean:

Tautologies in subset and partition logics: II

$$\begin{array}{ccc} U = \{*\} & & \mathbf{1} = \{\{0\},\{1\}\} \\ | & & | \\ \emptyset = \{\} & & \mathbf{0} = \{\{0,1\}\} \\ \wp(1) & \cong & \Pi(2) \end{array}$$

- **Theorem**: Every weak partition tautology is a subset tautology. *Proof*: If a formula is never assigned to $\mathbf{0}$ in $\Pi(2)$ then it is always assigned to $\mathbf{1}$ in $\Pi(2)$ and, by isomorphism, is always assigned to 1 in $\wp(1)$ so it is a subset tautology. $\square$
- **Corollary**: Every partition tautology is a subset tautology.

Partition tautologies neither included in nor include Intuitionistic tautologies

Notation: $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$ is π-negation & $\neg\sigma = \sigma \Rightarrow \mathbf{0}$.

Subset Tautologies	Intuit.	Partition	Weak Part.
$\sigma \Rightarrow (\pi \vee \sigma)$	Yes	Yes	Yes
$\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$	No	Yes	Yes
$\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))$	Yes	No	No
$\sigma \vee \overset{\pi}{\neg}\sigma$	No	No	Yes
$\tau \Rightarrow \left(\left(\tau \wedge \overset{\pi}{\neg}\sigma\right) \vee \left(\tau \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right)\right)$	No	No	No

Examples of subset, intuitionistic, partition, and weak partition tautologies.

Representation: partitions as binary relations

- Build representation of partition algebra $\Pi(U)$ using 'open' subsets of $U \times U$.
- Associate with partition π, the subset of distinctions made by π, $\text{dit}(\pi) = \{(u, u') : u \text{ and } u' \text{ in distinct blocks of } \pi\}$.
- *Closed subsets* of U^2 are reflexive-symmetric-transitive (rst) closed subsets, i.e., equivalence relations on U.
- *Open subsets* are complements, which are precisely dit-sets $\text{dit}(\pi)$ of partitions (= apartness relations in CompSci).
- For any $S \subseteq U \times U$, *closure* $cl(S)$ is rst closure of S.
- *Interior* $Int(S) = (cl(S^c))^c$ where $S^c = U \times U - S$ is complement.
- Closure op. *not* topological: $cl(S) \cup cl(T)$ not nec. closed, i.e., union of two equivalence relations is not nec. an eq. relation.

Partition op. = Apply set op. to dit-sets & take interior

Partition Op.	Representation
Join: $\sigma \vee \pi$	$\operatorname{dit}(\sigma \vee \pi) = \operatorname{dit}(\sigma) \cup \operatorname{dit}(\pi)$
Meet: $\sigma \wedge \pi$	$\operatorname{dit}(\sigma \wedge \pi) = int\left[\operatorname{dit}(\sigma) \cap \operatorname{dit}(\pi)\right]$
Implication: $\sigma \Rightarrow \pi$	$\operatorname{dit}(\sigma \Rightarrow \pi) = int\left[\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)\right]$
Top: $\mathbf{1} = \{\{u\} : u \in U\}$	$\operatorname{dit}(\mathbf{1}) = int\left[U \times U\right] = U \times U - \Delta U$
Bottom: $\mathbf{0} = \{U\}$	$\operatorname{dit}(\mathbf{0}) = int\left[\emptyset\right] = \emptyset$

Representation of $\Pi(U)$ in $Open(U \times U)$ by $\pi \longmapsto \operatorname{dit}(\pi)$.

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THE LOGIC OF PARTITIONS: INTRODUCTION TO THE DUAL OF THE LOGIC OF SUBSETS

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Abstract. Modern categorical logic as well as the Kripke and topological models of intuitionistic logic suggest that the interpretation of ordinary "propositional" logic should in general be the logic of subsets of a given universe set. Partitions on a set are dual to subsets of a set in the sense of the category-theoretic duality of epimorphisms and monomorphisms—which is reflected in the duality between quotient objects and subobjects throughout algebra. If "propositional" logic is thus seen as the logic of subsets of a universe set, then the question naturally arises of a dual logic of partitions on a universe set. This paper is an introduction to that logic of partitions dual to classical subset logic. The paper goes from basic concepts up through the correctness and completeness theorems for a tableau system of partition logic.

An introduction to partition logic

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Abstract

Classical logic is usually interpreted as the logic of propositions. But from Boole's original development up to modern categorical logic, there has always been the alternative interpretation of classical logic as the logic of subsets of any given (non-empty) universe set. Partitions on a universe set are dual to subsets of a universe set in the sense of the reverse-the-arrows category-theoretic duality–which is reflected in the duality between quotient objects and subobjects throughout algebra. Hence the idea arises of a dual logic of partitions. That dual logic is described here. Partition logic is at the same mathematical level as subset logic since models for both are constructed from (partitions on or subsets of) arbitrary unstructured sets with no ordering relations, compatibility or accessibility relations, or topologies on the sets. Just as Boole developed logical finite probability theory as a quantitative treatment of subset logic, applying the analogous mathematical steps to partition logic yields a logical notion of entropy so that information theory can be refounded on partition logic. But the biggest application is that when partition logic and the accompanying logical information theory are 'lifted' to complex vector spaces, then the mathematical framework of quantum mechanics (QM) is obtained. Partition logic models the indefiniteness of QM while subset logic models the definiteness of classical physics. Hence partition logic may provide the backstory so the old idea of 'objective indefiniteness' in QM can be fleshed out to a full interpretation of quantum mechanics. In that case, QM will be the 'killer application' of partition logic.

Examples of basic open questions in partition logic

- A decision procedure for partition tautologies.
- A Hilbert-style axiom system for partition tautologies, plus a completeness proof for that axiom system.
- Finite-model property: If a formula is not a partition tautology, does there always exist a finite universe U and partitions on that set so that the formula does not evaluate to $\mathbf{1}$.

Logical Prob. dual to Logical Information

Normalized counting measures on elements & distinctions

	Logical Probability Theory	Logical Information Theory
'Outcomes'	Elements $u \in U$ finite	Distinctions $(u,u') \in U \times U$ finite
'Events'	Subsets $S \subseteq U$	Dit sets $\mathrm{dit}(\pi) \subseteq U \times U$
Normalized counting measure	$\mathrm{Prob}(S) = \lvert S\rvert / \lvert U\rvert$ = *logical probability* of event S	$h(\pi) = \lvert \mathrm{dit}(\pi)\rvert / \lvert U \times U\rvert$ = *logical entropy* of partition π
Interpretation equiprobable outcomes	$\mathrm{Prob}(S)$ = probability randomly drawn element is an outcome in S	$h(\pi)$ = probability randomly drawn pair (w/replacement) is a distinction of π

Logical information theory

- Basic idea: information = distinctions
- Normalized count of distinctions = Info. measure = logical entropy
- Progress of definition of logical entropy:
  - Logical entropy of partitions: $h(\pi) = \frac{|\operatorname{dit}(\pi)|}{|U \times U|} = 1 - \sum_{B \in \pi} \left(\frac{|B|}{|U|}\right)^2$;
  - Logical entropy of probability distributions: $h(p) = 1 - \sum_i p_i^2$;
  - Logical entropy of density operators: $h(\rho) = 1 - \operatorname{tr}\left[\rho^2\right]$ in quantum information theory.

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Counting distinctions: on the conceptual foundations of Shannon's information theory

David Ellerman

Abstract Categorical logic has shown that modern logic is essentially the logic of subsets (or "subobjects"). In "subset logic," predicates are modeled as subsets of a universe and a predicate applies to an individual if the individual is in the subset. Partitions are dual to subsets so there is a dual logic of partitions where a "distinction" [an ordered pair of distinct elements (u, u') from the universe U] is dual to an "element". A predicate modeled by a partition π on U would apply to a distinction if the pair of elements was distinguished by the partition π, i.e., if u and u' were in different blocks of π. Subset logic leads to finite probability theory by taking the (Laplacian) probability as the normalized size of each subset-event of a finite universe. The analogous step in the logic of partitions is to assign to a partition the number of distinctions made by a partition normalized by the total number of ordered $|U|^2$ pairs from the finite universe. That yields a notion of "logical entropy" for partitions and a "logical information theory." The logical theory directly counts the (normalized) number of distinctions in a partition while Shannon's theory gives the average number of binary partitions needed to make those same distinctions. Thus the logical theory is seen as providing a conceptual underpinning for Shannon's theory based on the logical notion of "distinctions."

Developing Logical Entropy in Quantum Information Theory

Logical Entropy for Quantum States

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(Dated: January 26, 2015)

A Holevo-type bound for a divergence distance measure

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We find this framework of partitions and distinction most suitable (at least conceptually) for describing the problems of quantum state discrimination, quantum cryptography and in general, for discussing quantum channel capacity. In these problems, we are basically interested in a distance measure between such sets of states, and this is exactly the kind of knowledge provided by logical entropy [5]. In this work we shall focus on the basic definitions and properties and leave other advanced topics for future research [7].

Logical entropy in quantum measurement: I

- Density state ρ before measurement is a pure state. Three possible eigenstates each with probability $\frac{1}{3}$ = diagonal elements.
- But pure state is superposition of 3 eigenstates and off-diagonal elements given "coherences" between eigenstates.
- Since everything coheres together in pure state, $\rho^2 = \rho$ so $\mathrm{tr}\left[\rho^2\right] = 1$ and $h(\rho) = 1 - \mathrm{tr}\left[\rho^2\right] = 0$ since there are no distinctions = no information.

$$\rho(U) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \xrightarrow{measurement} \hat{\rho}(U) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Logical entropy in quantum measurement: II

- Non-degenerate measurement decoheres everything so all coherences vanish and these distinctions create the post-measurement information of $h(\hat{\rho}) = 1 - \operatorname{tr}\left[\hat{\rho}^2\right] = 1 - \frac{1}{3} = \frac{2}{3}$.

$$\rho(U) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \overset{measurement}{\longrightarrow} \hat{\rho}(U) = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

- Unlike von Neumann-Shannon, logical entropy shows exactly where the information comes from; the logical entropy created is the sum of all the coherences-squared that were zeroed-out, i.e., $6 \times \left(\frac{1}{3}\frac{1}{3}\right) = \frac{2}{3} = h(\hat{\rho})$.

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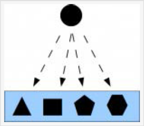
QUANTUM MECHANICS MATHEMATICS

On the Objective Indefiniteness Interpretation of Quantum Mechanics



Classical physics and quantum physics suggest two different meta-physical conceptions of reality: the classical notion of a objectively definite reality "all the way down," and the quantum conception of an objectively indefinite type of reality. Part of the problem of interpreting quantum mechanics (QM) is the problem of making sense out of an objectively indefinite reality. Our sense-making strategy is to follow the math by showing that the mathematical way to describe indefiniteness is by partitions (quotient sets or equivalence relations).

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