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NEW RESULTS ON THE STRAIGHT LINE AND HOSKOLD METHODS OF CAPITALIZATION

by David Ellerman, PhD

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INTRODUCTION

We show how the straight line (or Ring) and Hoskold methods of capitalizing income streams can be seen as the normal discounting of certain declining income streams where the decline in income can be conceptualized as interest losses. These losses result, as it were, from a make-believe reinvestment of a capital recovery portion of the income in a hypothetical sinking fund with an interest rate below the discount rate (0 in the straight line case and some "safe" rate i_s in the Hoskold case). We use a general result about amortization tables to prove that the discounted present values of the declining income streams are the values obtained by the straight line and Hoskold methods.

To fix notation, let a(n,i) be the annuity factor that is the present value of n $1 payments at the end of each year for n years, so that 1/a(n,i) is the installment payment at the end of each of the first n years that would amortize a loan of $1. Let s(n,i) be the accumulated future value at the end of the n^{th} year of n $1 deposits in a sinking fund so that 1/s(n,i) = SFF(n,i) is the yearly sinking fund deposit that would accumulate to $1 at the end of year n (where the sinking fund accumulates at the interest rate of i). Unless otherwise noted, we will use the abbreviation $a_n = a(n,i)$ and $s_n = s(n,i)$.

THE STRAIGHT LINE AND HOSKOLD CAPITALIZATION RATES

There is some controversy in the field of real estate appraisal over the status of the so-called "straight line" method (also called "Ring" method) and the Hoskold method of determining direct capitalization rates.

Method to Determine Capitalization Rate	Return of Investment	+ Return on Investment	= Capitalization Rate R
Straight Line Method	SFF (n,0)	i	i + 1/n
Hoskold Method @ i_s	SFF (n, i_s)	i	i + SFF (n, i_s)
Annuity Method @ i	SFF (n, i)	i	1/a(n,i)

We will show that the straight line and Hoskold capitalization rates will, when divided into the first year's income, give the correct present value only for certain declining income streams. Today's standard texts recognize that the straight line or Ring method is appropriate for certain declining income streams, but that recognition seems to be absent in the case of the Hoskold method.

SINKING FUNDS: REAL AND HYPOTHETICAL

The annuity or Inwood formula is based on the following equation of financial mathematics:

$$\frac{1}{a(n,i)} = i + \frac{1}{s(n,i)} = i + SFF(n,i).$$

Annuity Formula

This formula can be justified by considering several ways to pay off a loan of $1 in n payments. One method is to pay off the interest of i each year for n years. At the end of year n, the final interest payment of i is made and an additional balloon payment of the principal of $1 is made. A variant is to make the annual interest payments of i and to also make the annual sinking fund deposits of $1/s(n,i)$ which would accumulate to the balloon payment of $1. That method shows that a series of n equal payments of $i + 1/s(n,i)$ would pay off a loan of $1. But the series of payments equal to the installment to amortize one $1/a(n,i)$ will also pay off the loan so the two payments must be equal—which yields the annuity formula.

There seems to be much loose discussion of "sinking funds" in the real estate literature without any clear assumption about the sinking funds being actual or only "heuristic" (i.e., only "as if") and without any clear reason why the sinking fund needs to be coupled together with the income-producing property as a composite investment. The above argument refers to a sinking fund only as a heuristic device. The annuity formula is a mathematical relationship that holds independently of any assumption about the existence of an actual sinking fund. The formula can be proved directly with algebra.

Given a series of incomes generated by an income-producing property, the series has a present value (determined by the discount rate i) quite independently of the question of whether or not the incomes are reinvested in a sinking fund or in any other investment opportunity. When one assumes that the capital recovery part of the income is reinvested in a sinking fund with a certain interest rate and then considers the combined result, then one is analyzing a new composite investment: the income property plus the sinking fund. Consideration of a composite investment might be appropriate but it should be done explicitly and for some good reason.

Most references to "sinking funds" (e.g., in the discussion of the Hoskold method) seem to be without any clear assumption about an actual sinking fund or about why an actual sinking fund should be coupled with the income property as a composite investment. Why not consider an investment in some other real property or (say) in the Thailand stock market? If the analyst claims that no investment or sinking fund is available at the interest rate i, then this may simply be an argument that the assumed discount rate should be lower. In that case, a lower discount rate should be used straight-away rather than confusing the situation with reference to hypothetical sub-standard sinking funds at a "safe interest rate."

Another confusion common in discussions of the Hoskold method is the substitution of some extraordinary rate of return on, say, a coal mine or gold mine, for the interest rate as the discount rate in the Hoskold formula $R = i + SFF(n,i_s)$. Discussion of a rate of return on the investment is not appropriate since that utilizes some cost figure for the investment (which is irrelevant to an income determination of value like the Hoskold method) or it already assumes a given value for the investment. Thus it is bogus to reason that since one probably cannot find the same high rate of return for the capital recovery reinvestment, one must use a lower safe rate i_s for the capital recovery sinking fund. The i in the formula is the discount rate, not some presumed extraordinary rate of return, and if one cannot transform present money into future money at the rate i, then a lower value of i should be chosen as the discount rate.

MOTIVATION OF THE STRAIGHT LINE CAPITALIZATION FORMULA

We will show that the straight line formula (as well as the Hoskold formula) applies to certain declining income streams from a property (without any reference to a sinking fund). Sinking funds are relevant as a heuristic device because one can "motivate" the declining income stream as the combined income stream yielded by the composite investment of an income property giving a level income stream plus a sinking fund with a sub-standard interest rate. The decline in the total or composite income stream is precisely equal to the interest rate losses due to the reinvestment at a substandard interest rate.

Consider a declining income stream with I as the first year's income which then declines by the amount h each year for n years. The present value of the income stream at the discount rate i is:

$$V = \frac{I}{(1+i)^1} + \frac{I-h}{(1+i)^2} + \frac{I-2h}{(1+i)^3} + \cdots + \frac{I-(n-1)h}{(1+i)^n}.$$

We consider the hypothetical composite investment consisting of an income property with level income I and reinvestment of the capital recovery portion of income in a "mattress sinking fund." Suppose that the income from only the property is constant amount I for n years. At the end of each year part of the proceeds are reinvested in a sinking fund at the ultra-safe or "mattress" interest rate of zero. The value of the composite investment, property plus sinking fund, is V. At the end of each year, $SFF(n,0)V = V/n$ is invested in the zero-interest sinking fund. Thus at the end of second year, there is an interest loss of $h = iV/n$. At the end of each subsequent year, there is an additional loss of $h = iV/n$. Thus the combined income stream has the following present value where $h = iV/n$.

$$V = \frac{I}{(1+i)^1} + \frac{I-\left(iV/n\right)}{(1+i)^2} + \frac{I-2\left(iV/n\right)}{(1+i)^3} + \cdots + \frac{I-(n-1)\left(iV/n\right)}{(1+i)^n}.$$

This value could be found using the formula for valuing the straight line (constant amount) changing annuity but that formula does not work for the declining income stream obtained in the Hoskold case. Hence we will later present another method using generalized amortization tables that will work for both income streams and is more intuitive. We will show that the above value of the declining income stream is the same value as obtained by the straight line formula:

$$V = \frac{I}{i + \frac{1}{n}} = \frac{I}{i + SFF(n,0)}.$$

Straight Line Capitalization Formula

Thus the specific declining income stream appropriate for the straight line formula can be motivated as the composite result of a constant income stream plus reinvestment of part of the proceeds each year in a mattress sinking fund. It is unlikely that an appraiser will be asked to appraise the composite investment of a level income property plus a mattress sinking fund. Thus it is easy to see that the sinking fund in this case is only a heuristic or hypothetical device to motivate the decline in the income stream "as if" they were the interest losses from a mattress sinking fund. The sinking fund is just as hypothetical in the Hoskold case.

THE HOSKOLD FORMULA

Let i_s be the "safe" interest rate (between 0 and i) on our hypothetical sinking fund. Given a level income I on our property, the net income from our composite investment is I minus the sinking fund losses. Since the sinking fund losses compound as time passes, the income stream declines, and declines at an increasing rate.

To arrive at the specific declining income stream for the Hoskold case, we must find the interest loss resulting from investing in the sub-standard sinking fund at the safe rate i_s. We will use the sinking fund factors $SFF(k,i_s)$ and accumulations of 1 per period $s(k,i_s)$ at the safe rate i_s.

$$s(k,i_s) = (1+i_s)^{k-1} + \cdots + (1+i_s)^1 + 1 = \frac{(1+i_s)^k - 1}{i_s} = \frac{1}{SFF(k,i_s)}$$

In our safe sinking fund, we must invest at the end of each year for n years the amount that will accumulate to V, and that amount is $SFF(n,i_s)V$. After that amount is invested at the end of year 1, the interest rate loss at the end of year 2 from investing in the substandard sinking fund is $(i-i_s)SFF(n,i_s)V$.

At the end of year 3, there is the same loss on the amount invested at the end of year 2 but there is also the loss of what would have been the sinking fund accumulation on the previous loss. Thus the loss at the end of year 3 is

$$[\,(1 + i_s) + 1\,]\,(i - i_s)\,SFF\,(n, i_s)\,V = (i - i_s)\,SFF\,(n, i_s)\,Vs\,(2, i_s).$$

By similar reasoning we see that the loss at the end of year k+1 is

$$(i - i_s)\,SFF\,(n,i_s)\,Vs\,(k,i_s).$$

Interest Losses in k^{th} Year

Thus the declining income stream for the Hoskold case is:

$$I_k = I - (i - i_s)\,SFF\,(n, i_s)\,Vs\,(k, i_s).$$

Declining Income Stream in Hoskold Case

We will later use the method of general amortization tables to prove that the discounted value of that declining income stream is the usual value obtained using the Hoskold method of direct capitalization.

$$V = \frac{I}{i + SFF(n, i_s)}\ .$$

The Hoskold Formula

A GENERALIZED AMORTIZATION TABLE

The assertion about the Hoskold can be proven directly using the language of algebra. Since not all appraisers are fluent in that language, it might be more persuasive to restate the principal results using amortization tables. This requires a more general amortization table where the "principal reduction" or "capital recovery" can take place at a rate r not necessarily the same as the discount rate i. When $r = 0$, we will have an amortization table for the straight line or Ring method which shows the declining income for that case. When $r = i_s$ between 0 and i, we have a Hoskold amortization table that shows the declining income for that case. When $r = i$, we have usual amortization table with level income or amortization payments. If $r > i$, we have an amortization table with involves capital recovery at a supra-standard rate r and which thus generates a rising income stream.

We begin with a general theorem about amortization tables where the n principal reductions $P_1, P_2, \ldots, P_n$ can be chosen arbitrarily. Then we specialize to a situation where the principal is being reduced or capital is being recovered by a sinking fund that accumulates at a rate r not necessarily the same as the discount rate i. The principal or capital to be recovered is defined as the sum of those given principal reductions. Certain relationships hold between the columns in an amortization table. The interest in each year is the rate i times the balance or unrecovered capital from the previous year. The entry in the payment or income column is the sum of the interest and principal reduction (or capital recovery) columns. The entry in the balance (or unrecovered capital) column is the previous entry in the column minus the principal reduction (or capital recovery). The last entry in the balance or unrecovered capital column is zero.

Let $P_1, P_2, \ldots, P_n$ be the given principal reductions, let $V = P_1+P_2+\ldots+P_n$ be the sum, and let i be the discount rate. That is the only data given for the following general theorem about amortization tables.

General Amortization Table

Year	Income	= Interest +	Principal Reduction	Balance
1	$I_1 = P_1+i(P_1+\ldots+P_n)$	iV	P_1	$V-P_1$
2	$I_2 = P_2+i(P_2+\ldots+P_n)$	$i(V-P_1)$	P_2	$V-P_1-P_2$
...	...	...	...	...
k	$I_k = P_k+i(P_k+\ldots+P_n)$	$i(V-P_1-\ldots-P_{k-1})$	P_k	$V-P_1-\ldots-P_k$
...	...	...	...	...
n	$I_n = P_n + iP_n$	$i(V-P_1-\ldots-P_{n-1})$	P_n	$V-\Sigma P_k = 0$
			$\Sigma P_k = V$	

The other columns are all defined in terms of the given P_i's in the manner indicated. The incomes I_k's are determined as the sum of the Interest and Principal Reduction columns, and the general formula is

$$I_k = P_k + i(P_k + \ldots + P_n).$$

The Main Theorem is that the discounted present value of these incomes is the value V, the sum of the arbitrarily given P_k's.

$$\sum_{k=1}^{n} \frac{I_k}{(1+i)^k} = \sum_{k=1}^{n} P_k$$

Main Theorem on Amortization Tables

A proof is given in the Appendix.

AMORTIZATION TABLES WITH SINKING FUND CAPITAL RECOVERY

Let V be the value of the investment (or loan) and n the number of years to recover the capital (or pay off the loan). Let i be the interest rate and r be the rate for the capital recovery sinking fund. The value of the first year's income (or payment) is I. The value V is related to the first year's income by the direct capitalization formula:

$$V = \frac{I}{i + SFF\,(n,r)}.$$

The new deposit in the sinking fund each year to recover the capital is SFF(n,r)V which is abbreviated SFFV. After the deposit at the end of the k^{th} year, the amount in the sinking fund is SFFVs(k,r) which abbreviated $SFFVs_k$ Therefore the capital recovery during the k^{th} year due to both the new deposit and the new interest is $SFFVs_k - SFFVs_{k-1} = SFFV(1+r)^{k-1}$ and that is the entry in the k^{th} row of the capital recovery (or principal reduction) column. Each year's income I_k beginning with $I_1 = I$ is the sum of the interest (or return on unrecovered capital) and the capital recovered (return of capital) for that year.

Amortization Table with Sinking Fund Capital Recovery

Year	Income	= Interest +	Capital Recovered	Balance
1	I	iV	SFFV	V(1-SFF)
2	I_2	iV(1-SFF)	SFFV(1+r)	$V(1-SFFs_2)$
3	I_3	$iV(1-SFFs_2)$	$SFFV(1+r)^2$	$V(1-SFFs_3)$
...	...	...	...	...
n	I_n	$iV(1-SFFs_{n-1})$	$SFFV(1+r)^{n-1}$	$V(1-SFFs_n)$

Since SFF = $1/s_n$ the last entry in the Balance or Unrecovered Capital column is 0. The sum of the Capital Recovered column is

$$SFFV + SFFV\,(1 + r) + SFFV\,(L + r)^2 + \ldots + SFFV\,(1 + r)^{n-1}$$
$$= SFFVs_n = V$$

as desired. The incomes I_k are obtained as the sum of the Interest and Capital Recovered columns. It is useful to compute the first few incomes.

$$I_2 = i\,V\,(1 - SFF) + SFFV\,(1{+}r)$$
$$= i\,V + SFFV - iSFFV + rSFFV$$
$$= I - (i - r)SFFV$$

The income for the 2nd year is I minus (i-r)SFFV which is the interest loss on the sinking fund deposit of SFFV.

The third year's income is calculated as follows.

$$I_3 = iV\,(1\text{-}SSFs_2) + SFFV\,(1 + r)_2$$
$$= iV - iVSFF + SFFV\,(1 + r) - (i - r)\,SFFV\,(1 + r)$$
$$= I_2 - (i - r)\,SFFV\,(1 + r)$$
$$= I - (i - r)\,SFFVs_2.$$

Thus we see that each year's income I_k is I minus the interest losses on the sinking fund (assuming $r<i$) where the latter can be calculated as $(i\text{-}r)SFFVs_k$, the accumulation s_k on the interest losses (i-r) on the sinking fund deposits SFFV:

$$I_k = I - (i - r)SFFVs_k.$$

Since these incomes I_k are the same as those obtained in our previous analysis of the Hoskold case when $r = i_s$, the Main Theorem on Amortization Tables gives the proof that the present value of these incomes is the value $V = I/[i{+}SFF(n,r)]$.

In the straight line or Ring case or $r = 0$, SFF = $1/n$ and $s_k = k$ so the declining income is given by $I_k = I - i(V/n)k$ as previously obtained in our analysis of the straight line case. Thus the Main Theorem on Amortization Tables also shows that the present value of that declining income stream is the value obtained by the straight line method.

In the straight line case, the income stream declines by a constant amount iV/n each year independent of k. In the Hoskold case, the drop in the income stream from I_k to I_{k+1} is $(i-r)SFFV(s_{k+1} - s_k) = (i-r)SFFV(1{+}r)^k$ which depends on k. The drop in the income stream in each period is (1+r) times the previous drop. This is illustrated in the following amortization table based on the Hoskold situation where $0 < r < i$. The change in the incomes accelerates at the sinking fund rate of r (as seen in the right-most column of the following spreadsheet).

Amortization Table with Sinking Fund Capital Recovery: Hoskold Case

1st Income =	100.00		n = 5			
i =	10%	= Disc. Rate	V = 355.90			
r =	5%	= Sinking Fund Rate				
						% Change in
Year	Income	Interest	Capital Recovery	Balance	ΔI	ΔI
1	100.00	35.59	64.41	291.49		
2	96.78	29.15	67.63	223.86	3.2205	
3	93.40	22.39	71.01	152.85	3.3815	5.00%
4	89.85	15.29	74.56	78.29	3.5506	5.00%
5	86.12	7.83	78.29	0.00	3.7281	5.00%
		Sum =	355.90			

In the straight line or Ring case, we set the sinking fund rate equal to 0.

Amortization Table with Sinking Fund Capital Recovery: Straight Line Case

1st Income =	100.00		n = 5			
i =	10%	= Disc. Rate	V = 333.33			
r =	0%	= Sinking Fund Rate				
						% Change in
Year	Income	Interest	Capital Recovery	Balance	ΔI	ΔI
1	100.00	33.33	66.67	266.67		
2	93.33	26.67	66.67	200.00	6.6667	
3	86.67	20.00	66.67	133.33	6.6667	0.00%
4	80.00	13.33	66.67	66.67	6.6667	0.00%
5	73.33	6.67	66.67	0.00	6.6667	0.00%
		Sum =	333.33			

When r = i, we have an ordinary amortization table where i—r = 0 so the interest loss is 0 and the income is constant.

Amortization Table with Sinking Fund Capital Recovery: Ordinary Case r = 1

1st Income =	100.00		n = 5			
i =	10%	= Disc. Rate	V = 379.08			
r =	10%	= Sinking Fund Rate				
Year	Income	Interest	Capital Recovery	Balance	ΔI	
1	100.00	37.91	62.09	316.99		
2	100.00	31.70	68.30	248.69	0.0000	
3	100.00	24.87	75.13	173.55	0.0000	
4	100.00	17.36	82.64	90.91	0.0000	
5	100.00	9.09	90.91	0.00	0.0000	
		Sum =	379.08			

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NEW RESULTS ON THE STRAIGHT LINE AND HOSKOLD METHODS OF CAPITALIZATION

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SUMMARY ON THE STRAIGHT LINE AND HOSKOLD METHODS

We have seen that the straight line and Hoskold formulas can each be derived in two ways:

(1) as the value of the composite income stream resulting from an income property with a constant income I plus an actual sinking fund to recover the value at a substandard interest rate, and

(2) as the value of a declining income stream where each year's decline in income can be heuristically thought of as the interest losses resulting from reinvestment in a hypothetical sinking fund to recover the value at a substandard interest rate.

In the straight line case, the substandard interest rate is 0, and in the Hoskold case, the substandard safe rate is in between 0 and the discount rate i.

The two methods are treated differently in the usual textbooks—as if the income stream was declining for the straight line case but was constant for the Hoskold case. It is straightforward to see that the straight line case can be treated in parallel to the Hoskold case—as the composite result of a constant income stream from the property coupled together with the recovery of the capital in the "mattress sinking fund" with a zero interest rate.

The more difficult result derived here is that the Hoskold case can be treated in parallel to the straight line case—as the value of a declining income stream where the income decline is motivated as the interest losses from trying to recover the investment in a hypothetical substandard sinking fund with a safe interest rate i_s.

It is a common fallacy in finance to think that the valuation of an income stream from an investment requires some assumption about the reinvestment of the income thrown off by the investment. But any reinvestment immediately couples the given investment with some new second investment in a composite investment. By this methodology, the valuation of the composite investment requires, in turn, assumptions about its income reinvestments, so we are quickly into an infinite regress. The fallacy occurred at the beginning; no assumptions are required about the subsequent fate of the incomes thrown off by the first investment. The investment can be valued by itself.

There seems to be no valid reason to consider an actual reinvestment in a mattress or substandard sinking fund when applying the straight line or Hoskold methods. Each method applies to a single investment in an income producing property with a particular type of declining income stream. If the particular shape of the decline can be considered as the interest losses in a hypothetical sinking fund at the rate of 0 or i_s, then (respectively) the straight line or Hoskold cap rates may be applied to the first year's income to determine the value. In the interest of realism, we must add that appraisers seem to use the straight line and Hoskold formulas (to the extent that they do) because of the simplicity of the formulas, not because of any serious estimation that the income streams decline in the particular manner that would justify those formulas as opposed to the infinity of other formulas available for declining income streams.

APPENDIX: PROOF OF THE MAIN THEOREM ON AMORTIZATION TABLES

To prove the result,

$$\sum_{k=1}^{n} \frac{I_k}{(1+i)^k} = \sum_{k=1}^{n} P_k$$

where $I_k = P_k + i(P_k + \ldots + P_n) = (1+i)P_k + i(P_{k+1} + \ldots + P_n)$ we need to evaluate the sum

$$\sum_{k=1}^{n} \frac{I_k}{(1+i)^k} = \sum_{k=1}^{n} \frac{(1+i)P_k + i(P_{k+1} + \cdots + P_n)}{(1+i)^k}.$$

To rearrange the sum, we consider the following table of the terms to be discounted at t=1,2,...,n. Each row gives the income for that time period, the sum of the table entries across the row times the P_j's at the head of the columns.

	P_1	P_2	P_3	$\cdots$	P_k	$\cdots$	P_n
$t = 1{:}I_1$	$1+i$	i	i	$\cdots$	i	$\cdots$	i
$t = 2{:}I_2$		$1+i$	i	$\cdots$	i	$\cdots$	i
$\vdots$			$\ddots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$
$t = k{:}I_{k-1}$				$1+i$	i	$\cdots$	i
$t = k{:}I_k$					$1+i$	$\cdots$	i
$\vdots$						$\ddots$	$\vdots$
$t = k{:}I_n$							$1+i$

We can now easily rewrite the sum as the discounted present value of the entries in the columns to obtain:

$$\sum_{k=1}^{n} \frac{I_k}{(1+i)^k} = \sum_{k=1}^{n} P_k\left(\frac{1}{(1+i)^k} + i\left[\frac{1}{(1+i)^k} + \cdots + \frac{1}{(1+i)^1}\right]\right) = \sum P_k\left(\frac{1}{(1+i)^k} + ia_k\right)$$

$$= \sum_{k=1}^{n} P_k\left(\frac{1}{(1+i)^k} + 1 - \frac{1}{(1+i)^k}\right) = \sum_{k=1}^{n} P_k$$

which completes the proof of the Main Theorem on Amortization Tables.

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