The Objective Indefiniteness Interpretation of Quantum Mechanics:

Partition Logic, Logical Information Theory, and Quantum Mechanics

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Outline of the argument

- Common-sense view of reality is expressed logically in Boolean subset logic—each element definitely in or not in a subset (property)—and does *not* work in QM.
- But a dual form of logic, partition logic (subsets and partitions are mathematically dual), has been developed.
- Basic idea: if there are two dual forms of logic, and one form does not work for QM, then try the other form, partition logic.
- When this partition math is "lifted" to vector spaces, it works! It indeed gives the math and relationships of quantum mechanics.
- Thus the vision of micro-reality described by partition logic indeed seems to be the micro-reality described by QM.
- Key concept is old idea of objective indefiniteness. Partition logic (including logical information theory) and lifting program provide the back story so that old idea then gives the *objective indefiniteness interpretation of QM*.

"Propositional" logic ⇒ Subset logic

- "*The algebra of logic has its beginning in 1847, in the publications of Boole and De Morgan. This concerned itself at first with an algebra or calculus of classes,… a true propositional calculus perhaps first appeared…in 1877.*" [Alonzo Church 1956]
- Variables refer to subsets of some universe U and operations are subset operations.
- Valid formula ("tautology") = result of substituting any subsets for variables is the universe set U for any U.
- Boole himself noted that to determine valid formulas, it suffices to only take subsets Ø = 0 and U = 1, which was later generalized by Renyi to probability theory.

Duality of Subsets and Quotient Sets

- Tragedy of 'propositional' logic; props don't dualize.
- Subsets have a dual (unlike propositions).
- Category Theory (CT) duality:
  - duality between injective and surjective set maps.
  - duality in algebra between subobjects & quotient objects, e.g., subgroups and quotient groups.
- CT duality gives subset-partition duality:
  - Injective map gives a subset of codomain (its image);
  - Surjective map determines a partition of domain (its inverse-image).
  - "*The dual notion (obtained by reversing the arrows) of 'part'* [subobject] *is the notion of partition.*" (Lawvere)
  - A *set partition* of a set U is a collection of subsets $\pi = \{B, B', \ldots\}$ that are mutually disjoint and the union is U.

Two Lattices: Subsets and Partitions

- Given universe set U, there is the *Boolean lattice of subsets* $\wp(U)$ with inclusion as partial ordering and the usual union and intersection, and enriched with implication: $A \Rightarrow B = A^c \cup B$.
- Given universe set U, there is the *lattice of partitions* $\Pi(U)$ enriched by implication where refinement is the partial ordering.
  - Given partitions $\pi = \{B, B', \ldots\}$ and $\sigma = \{C, C', \ldots\}$, σ is *refined* by π, $\sigma \leq \pi$, if for every block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$.
  - *Join* $\pi \vee \sigma$ of $\pi = \{B, \ldots\}$ and $\sigma = \{C, \ldots\}$ is partition whose blocks are non-empty intersections $B \cap C$.
  - *Meet* $\pi \wedge \sigma$: define undirected graph on U with link between u and u' if they are in same block of π or σ. Then connected components of graph are blocks of meet.
  - *Implication* $\sigma \Rightarrow \pi$ is the partition that is like π except that any block $B \in \pi$ contained in some block $C \in \sigma$ is discretized (replaced by singletons).
  - *Top* = $\mathbf{1} = \{\{u\} | u \in U\}$ discrete partition of singletons; *Bottom* = $\mathbf{0} = \{U\}$ = indiscrete partition = "blob"

Table of Logic Dualities

	Subset Logic	**Partition Logic**
'Elements'	Elements $u \in U$	Distinctions $(u,u') \in (U \times U) - \Delta_U$
All 'elements'	Universe set U	Discrete partition **1** (all dits)
No 'elements'	Empty set $\emptyset$	Indiscrete partition **0** (no dits)
Variables in formulas	Subset $S \subseteq U$	Partition π on U
Interpretation	$f:S' \rightarrow U$ so $\mathrm{Im}(S') = S$ defines *property* on U.	$f:U \rightarrow R$ so $f^{-1}(R) = \pi$ defines R-valued *attribute* on U.
Logical operations	Subset ops $\cup$, $\cap$, $\Rightarrow$,…	Partition ops $\cong$ Interior of subset ops applied to dit-sets.
Formula $\Phi(\pi, \sigma,\ldots)$ holds of an 'element'	Element u is in $\Phi(\pi,\sigma,\ldots)$ as a subset.	A dit (u,u') is distinguished by $\Phi(\pi,\sigma,\ldots)$ as a partition.
Valid formula $\Phi(\pi,\sigma,\ldots)$	$\Phi(\pi,\sigma,\ldots) = U$ (top) for any subsets $\pi,\sigma,\ldots$ of any U $(1 \leq \|U\|)$.	$\Phi(\pi,\sigma,\ldots) = \mathbf{1}$ (top = discrete partition) for any partitions $\pi,\sigma,\ldots$ on any U $(2 \leq \|U\|)$.

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THE LOGIC OF PARTITIONS: INTRODUCTION TO THE DUAL OF THE LOGIC OF SUBSETS

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Abstract. Modern categorical logic as well as the Kripke and topological models of intuitionistic logic suggest that the interpretation of ordinary "propositional" logic should in general be the logic of subsets of a given universe set. Partitions on a set are dual to subsets of a set in the sense of the category-theoretic duality of epimorphisms and monomorphisms—which is reflected in the duality between quotient objects and subobjects throughout algebra. If "propositional" logic is thus seen as the logic of subsets of a universe set, then the question naturally arises of a dual logic of partitions on a universe set. This paper is an introduction to that logic of partitions dual to classical subset logic. The paper goes from basic concepts up through the correctness and completeness theorems for a tableau system of partition logic.

Why previous attempts to dualize logic were stymied

- Subset logic is typically identified with special case of propositional logic, and propositions, unlike subsets, do not have a dual so the very idea of a dual logic—dual to subset logic—was not "in the air."
- So-called "lattice of partitions" was written upside-down as the lattice of equivalence relations so basic analogy, "elements of subset" ≅ "distinctions of partition" was missed.
- As was noted in 2001, "*the only operations on the family of equivalence relations fully studied, understood and deployed are the binary join* $\vee$ *and meet* $\wedge$ *operations*." In particular, the crucial operation of implication was not defined, and the various algorithms to define partition ops from subset ops (e.g., closure space or graph theory) don't seem to have been studied.

Information Theory is to Partition Logic as Probability Theory is to Subset Logic

	Logical Finite Prob. Theory	Logical Information Theory
'Outcomes'	Elements $u \in U$ finite	Pairs $(u,u') \in U \times U$ finite, $u \neq u'$
'Events'	Subsets $A \subseteq U$	Distinctions of π, i.e., $\text{dit}(\pi) \subseteq U \times U$
Normalized size	$\text{Prob}(A) = \|A\|/\|U\|$ = number of elements (normalized).	$h(\pi) = \|\text{dit}(\pi)\|/\|U \times U\|$ = *Logical Entropy* of partition π = number of distinctions of π (normalized).
Equiprobable outcomes	$\text{Prob}(A)$ = probability randomly drawn element is in subset A	$h(\pi)$ = probability randomly drawn pair (w/replacement) is distinguished by partition π

- $\text{dit}(\pi)$ = set of distinctions [pairs (u,u') in different blocks] of π .
- Progress of definition of logical entropy:
  - Partitions: $h(\pi) = |\text{dit}(\pi)|/|U \times U| = 1 - \Sigma_{B \in \pi}[|B|/|U|]^2$;
  - Probability distributions: $h(p) = 1 - \Sigma p_i^2$;
  - Density operators in QM: $h(\rho) = 1 - \text{tr}(\rho^2)$.

Two Ways to Measure All the Distinctions of a Partition

- *Shannon entropy* of π (base 2) = $H(\pi) = \Sigma_B p_B \log(1/p_B)$ = average number of equal binary partitions (bits) needed to make <u>all the distinctions of π</u>.
- *Logical entropy* of $\pi = h(\pi) = \Sigma_B p_B(1-p_B)$ = normalized count of <u>all the distinctions of π</u>.
- Common concept of ***distinctions*** of a partition is fundamental in logic of partitions, logical info. theory and, as we will see, in quantum mechanics.

Parallel Concepts for Shannon and Logical Entropies

	Shannon Entropy	Logical Entropy
Entropy	$H(\pi) = \Sigma_B p_B \log(1/p_B)$	$h(\pi) = \Sigma_B p_B(1-p_B)$
Mutual Information	$I(\pi;\sigma) = H(\pi)+H(\sigma)-H(\pi\vee\sigma)$	$mut(\pi;\sigma) = h(\pi)+h(\sigma)-h(\pi\vee\sigma)$
Independence	$I(\pi;\sigma) = 0$	$mut(\pi;\sigma) = h(\pi)h(\sigma)$
Cross entropy	$H(p\|\|q) = \Sigma p_i \log(1/q_i)$	$h(p\|\|q) = \Sigma p_i(1-q_i)$
Divergence	$D(p\|\|q) = H(p\|\|q)-H(p)$	$d(p\|\|q) = 2h(p\|\|q) - h(p) - h(q)$
Information Inequality	$D(p\|\|q) \geq 0$ with $= 0$ iff $p_i = q_i$ for all i.	$d(p\|\|q) \geq 0$ with $= 0$ iff $p_i = q_i$ for all i.

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Counting distinctions: on the conceptual foundations of Shannon's information theory

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Abstract Categorical logic has shown that modern logic is essentially the logic of subsets (or "subobjects"). In "subset logic," predicates are modeled as subsets of a universe and a predicate applies to an individual if the individual is in the subset. Partitions are dual to subsets so there is a dual logic of partitions where a "distinction" [an ordered pair of distinct elements (u, u') from the universe U] is dual to an "element". A predicate modeled by a partition π on U would apply to a distinction if the pair of elements was distinguished by the partition π, i.e., if u and u' were in different blocks of π. Subset logic leads to finite probability theory by taking the (Laplacian) probability as the normalized size of each subset-event of a finite universe. The analogous step in the logic of partitions is to assign to a partition the number of distinctions made by a partition normalized by the total number of ordered $|U|^2$ pairs from the finite universe. That yields a notion of "logical entropy" for partitions and a "logical information theory." The logical theory directly counts the (normalized) number of distinctions in a partition while Shannon's theory gives the average number of binary partitions needed to make those same distinctions. Thus the logical theory is seen as providing a conceptual underpinning for Shannon's theory based on the logical notion of "distinctions."

Keywords Information theory · Logic of partitions · Logical entropy · Shannon entropy

This paper is dedicated to the memory of Gian-Carlo Rota—mathematician, philosopher, mentor, and friend.

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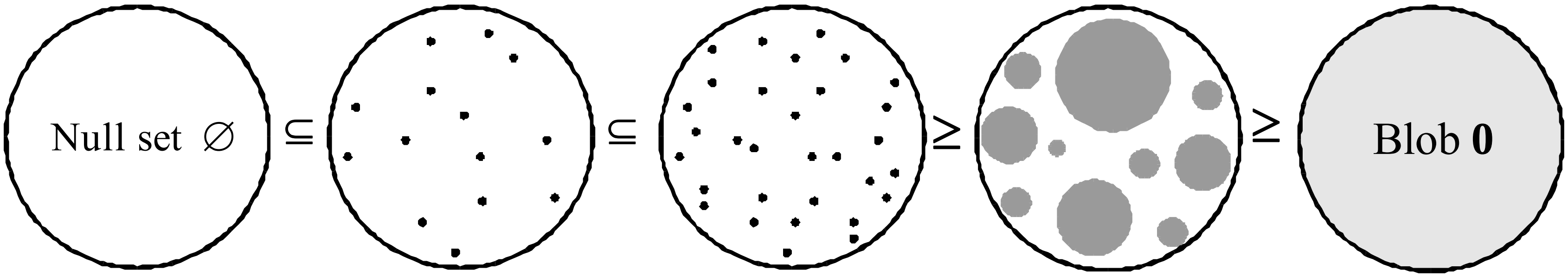
What in the world does Quantum Mechanics describe?

- Major conceptual problem left from 20th century physics is the interpretation of quantum mechanics.
- Common sense view of entities with a full set of properties ("properties all the way down") is described logically by the Boolean logic of subsets.
- But this view does *not* apply at quantum level.
- How can one find alternative visions of micro-reality?
- There is a unique dual to notion of subset, namely the notion of a quotient set or partition.
- Answer: Instead of subset logic, try the dual logic, partition logic.

The dual vision of micro-reality

- The new form of logic, partition logic, dual to ordinary Boolean subset logic, gives a new *vision* of micro-reality based on partitions and objectively indefinite entities, and that vision provides the *objective indefiniteness* interpretation of QM.
- **Basic idea**: interpret block of partition, say {a,b,c}, not as subset of three distinct elements; but as one *indistinct* element that—with distinctions—could be refined to {a}, {b}, or {c}.
- Overview of the argument here:
  - the mathematics of partitions using sets can be "lifted" to vector spaces,
  - the result is essentially the math of QM, and hence
  - the micro-reality described in QM *fits* this interpretation.

Dual creation stories: 2 ways to create a Universe *U*



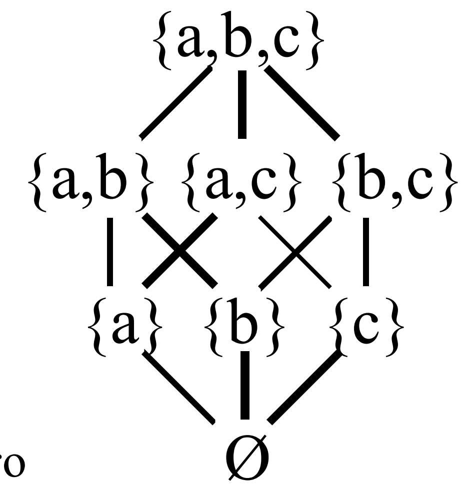
Subset creation story → **U** ← Partition creation story

- <u>Subset creation story</u>: "*In the Beginning was the Void*", and then elements are created, fully propertied and distinguished from one another, until finally reaching all the elements of the universe set *U*.
- <u>Partition creation myth</u>: "*In the Beginning was the Blob*", which is an undifferentiated "substance," and then there is a "Big Bang" where elements ("its") are created by being objectively in-formed (objective "dits") by the making of distinctions (e.g., breaking symmetries) until the result is finally the singletons which designate the elements of the universe *U*.
- In sum, to reach U from the beginning:
  - increase the size of subsets, or
  - increase the refinement of quotient sets.

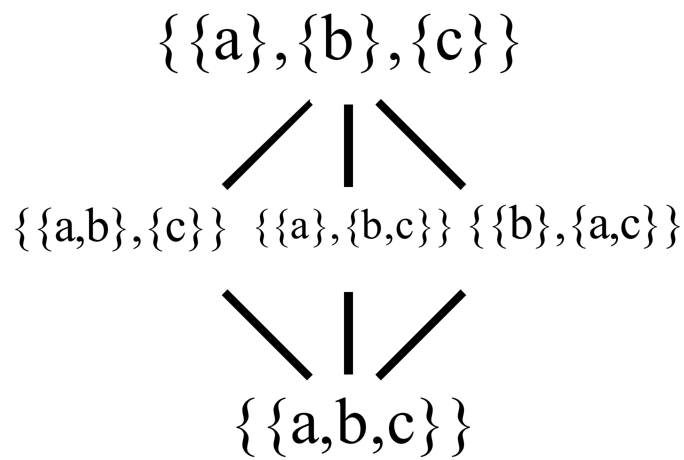
Conceptual duality between lattices

Substance increases, always fully formed.

Start with zero substance.



Subset lattice

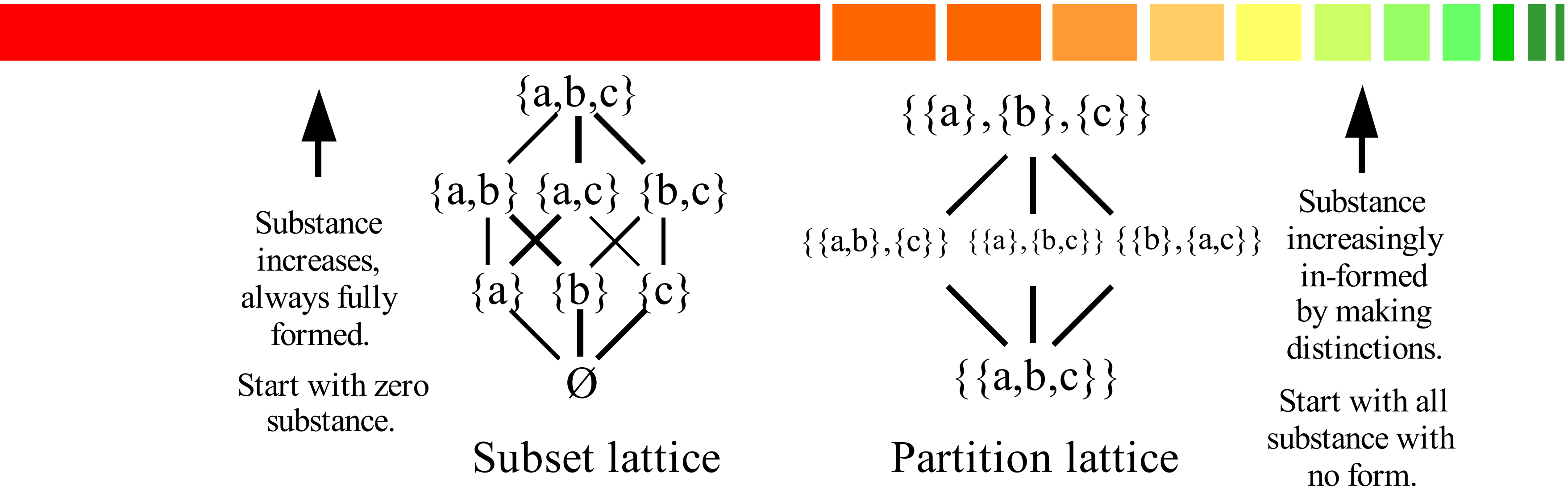


Partition lattice

Substance increasingly in-formed by making distinctions.

Start with all substance with no form.

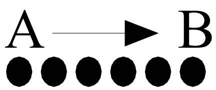
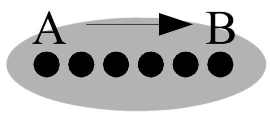
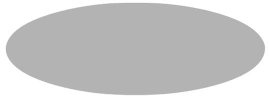
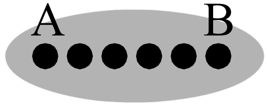
Heisenberg on substance (energy) & form



- Heisenberg: "Energy is a substance, since its total amount does not change, and the elementary particles can actually be made from this substance as is seen in many experiments on the creation of elementary particles."
- Heisenberg, in his rendering of Aristotle, refers to substance as: "a kind of indefinite corporeal substratum, embodying the possibility of passing over into actuality by means of the form."
- This in-forming is the making of distinctions, e.g., distinctions taking the actual indefinite state {a,b,c}, (where {a}, {b}, & {c} are only "potential") to, say, {a} or {b,c}, and then in the {b,c} case, with more distinctions, to say, {c}.

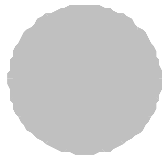
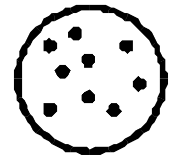
Old idea of objective indefiniteness

- Basic idea of *objective indefiniteness* is not new:
  - "objectively indefiniteness" emphasized particularly by Shimony.
  - "inherent indefiniteness" also mentioned by Feyerabend.
- As distinctions are made (measurements), *objectively indefinite or indistinct states* are made more distinct. Fully distinct states = *eigen* states (for some attribute).

Classical trajectory from A to B	
Subjective indefiniteness of classical position ("cloud of ignorance")	
"Quantum trajectory": like def. focus at A, going out of focus, & new def. focus at B	
Particle with objectively indefinite location…	
may be *represented* as superposition of possible eigen-positions.	

Superposition in pictures

Eigenstate 1: Guy Fawkes with goatee	
Eigenstate 2: Guy Fawkes with mustache	
Objectively indefinite state before (facial hair) distinctions were made is pre-distinction state.	
But—objectively indefinite state may be *represented* by math superposition of the possible distinct alternatives: $[\vert\text{goatee}> + \vert\text{mustache}>]/\sqrt{2}$.	

Indistinct entity  represented as  set (superposition) of possible eigen-alternatives.

- Mathematically, a state that is objectively indefinite between a number of possible fully distinct or eigen states is a weighted *vector sum* of the eigenstates.
- But this is ontologically misleading if taken to mean that the eigenstates already actual and are superposed—like a double exposure photograph—to arrive at the indefinite states (a hold over from classical wave thinking).
- Popular misconception that QM says particle is "here and there at the same time"; should be "not definitely here nor there."
- Superposition = the indefinite state before distinctions are made, where the eigenstates in the sum show what can result from the making of distinctions, i.e., measurement.

"Wave-particle duality" = Indefinite-definite particle complementarity

- Thus states that are indefinite for an observable are represented as weighted vector sums or “superpositions” of the fully-distinct eigen-alternatives.
- This indefiniteness-represented-by-superposition is seen as “wave-like aspects” of particles in indefinite state.
- Hence the distinction-making measurements take away the indefiniteness which is usually described as “collapse of the wave-packet.”
- But there are *no actual physical waves* in QM; *only particles with indistinct (or distinct) attributes*.
- Making distinctions gives “collapse of indefiniteness.”
- "Wave-particle duality" = "indefinite-definite" particle complementarity.

The Lifting Program: Sets → Vector Spaces

- Why vector spaces?
  - Because objective indefiniteness can be represented by superposition (vector sum) of fully distinct possible alternatives, and
  - Superposition basic to quantum mechanics (Dirac).
- It is part of math folklore that set concepts “lift” to vector space concepts.
- **Basis Principle**: Apply set concept to basis set and see what concept it generates in vector space.
- For instance, apply set concept of cardinality to basis set and get vector space concept of dimension. Cardinality lifts to dimension.
- The objective indefiniteness interpretation of QM is based on the Lifting Program to show how partition math for sets “lifts” to give the math of QM—and *thus* QM seems to describe the micro-reality suggested in the partition logic creation story.

What is the lift of a *set partition*?

- Concept of basis set is also the vehicle to lift concept of "set partition" to corresponding concept for vector spaces.
- Take a set partition $\pi = \{B, B', \ldots\}$ of a basis set; the blocks B generate subspaces $W_B \subseteq V$ which form a <u>direct sum decomposition</u> of V: $V = \sum_B \oplus W_B$.
- Hence a *vector space partition* is defined to be a direct sum decomposition of the space V, *not* a set partition of V.
- Some earlier proposition-oriented attempts to relate partitions to QM math failed by emphasizing that every subspace $W \subseteq V$ defines a *set partition* of V: $v \sim v'$ if $v - v' \in W$, i.e., wrong notion of partition in a vector space. "Quantum logic."

What is the lift of a *join* of set-partitions?

- Set Definition: Two set partitions $\pi = \{B, B', \ldots\}$ and $\sigma = \{C, C', \ldots\}$ are *compatible* if defined on a common universe U.
- Lifted Definition: Two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ are said to be *compatible* if they have a common basis set, i.e., if there is a basis set so they are generated by two set partitions on that same basis set.
- Set Definition: If two set partitions $\pi = \{B, B', \ldots\}$ and $\sigma = \{C, C', \ldots\}$ are compatible, their *join* $\pi \vee \sigma$ is defined and is the set partition whose blocks are the non-empty intersections $B \cap C$.
- Lifted Definition: If two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ are compatible, their *join* $\omega \vee \xi$ is defined and is the vector space partition whose blocks are the non-zero intersections $W_\lambda \cap X_\mu$ (which is generated by the join of the two set partitions on any common basis set).
- What is lift of a set-attribute $f: U \to \mathbb{R}$? [Not a functional since a functional determines a set partition (defined by its kernel), not a vector space partition.]

What is the lift of a *set-attribute* $f: U \to \mathbb{R}$?

- If f is constant on a subset $S \subseteq U$ with value r, then symbolize $f \restriction S = rS$, and call S an "eigenvector" of f and r an "eigenvalue."
- As subsets get smaller, all functions are eventually constant, so for any subset S, $\exists$partition $S_1,\ldots,S_n,\ldots$ of S such that $f \restriction S = r_1 S_1 + \ldots + r_n S_n + \ldots$.
- For any "eigenvalue" r, define $f^{-1}(r)$ = "eigenspace of r" as union of "eigenvectors" for that "eigenvalue."
- Since "eigenspaces" span U, function $f: U \to \mathbb{R}$ is represented by:

$$f = \sum_r r\chi_{f^{-1}(r)}.$$

"Spectral decomposition" of set attribute $f: U \to \mathbb{R}$.

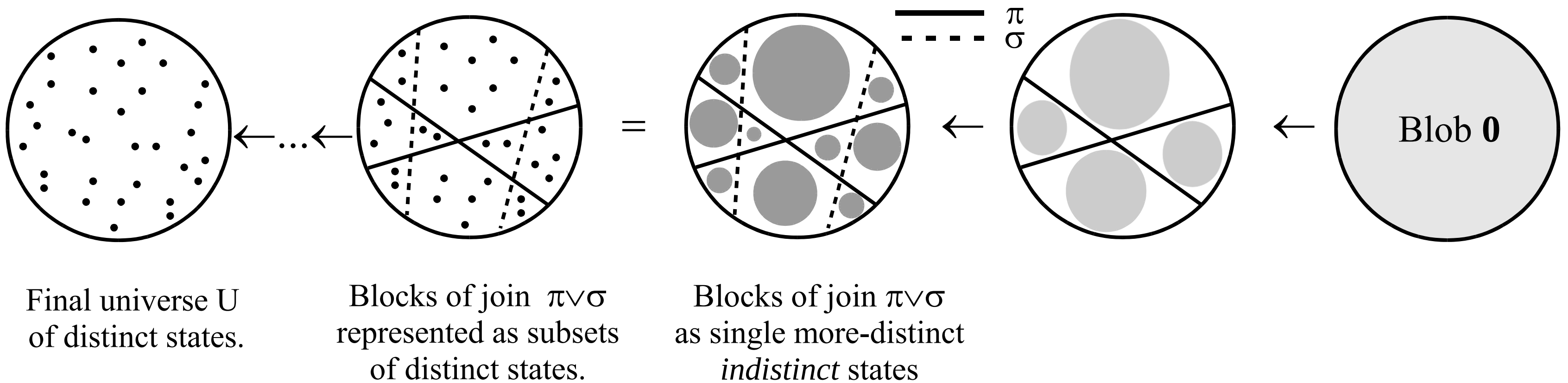
- Therefore an attribute, which is constant on blocks $\{f^{-1}(r)\}$ of a set partition, lifts to something constant on the blocks (subspaces) of a vector space partition. Spectral decomp.

$$f = \sum_r r\chi_{f^{-1}(r)} \text{ lifts to } L = \sum_\lambda \lambda P_\lambda.$$

Real-valued attributes lift to Hermitian linear operators!

Lift program:	Set concept	Vector space concept
Eigenvalues	r s.t. $f{\upharpoonright}S = rS$ for some S	λ s.t. $Lv = \lambda v$ for some v
Eigenvectors	S s.t. $f{\upharpoonright}S = rS$ for some r	v s.t. $Lv = \lambda v$ for some λ
Eigenspaces	$\cup\{S: f{\upharpoonright}S = rS\} = f^{-1}(r)$ for an "eigenvalue" r	$\{v: Lv = \lambda v\} = W_\lambda$ for an eigenvalue λ
Partition	Set partition of "Eigenspaces" $f^{-1}(r)$	Vector space partition of Eigenspaces W_λ
Characteristic functions	$\chi_S: U \to \{0,1\}$ for subsets S like $f^{-1}(r)$	Projection operators for subspaces like $W_\lambda = P_\lambda(V)$
Spectral decomposition	Set attribute $f: U \to \mathbb{R}$ $f = \Sigma_r r\chi_{f^{-1}(r)}$	Hermitian linear operator $L = \Sigma_\lambda \lambda P_\lambda$

Join of compatible partitions lifts to join of eigenspaces of commuting operators



- Set Fact: Join of inverse image partitions of two attributes defined iff attributes are compatible, i.e., have same domain.
- Lifted Fact: Eigenspace partitions of two linear ops are compatible so that join is defined iff the operators commute.
- Given two commuting operators L,M, eigenspace partitions are compatible so can take join, and vectors in blocks of join are simultaneous eigenvectors of the operators.

Complete joins determine eigen-alternatives

- Set Fact: Given two same-domain set attributes $f,g:U\rightarrow\mathbb{R}$, the blocks $f^{-1}(r)\cap g^{-1}(s)$ in the join are uniquely labeled by ordered pairs (r,s) of values, e.g., (age, weight) of people in a room.
- Set Definition: Set of same-set attributes is *complete* if join of their partitions is discrete (i.e., all 1-element subsets).
- Does this partition math carving out elements of U lift to a QM way to carve out eigenvectors? Yes.
- Lifted Fact: Given two commuting ops., the blocks $W_\lambda\cap X_\mu$ in the join of eigenspace partitions are uniquely labeled by ordered pairs of eigenvalues (λ,μ).
- Lifted Definition: Set of commuting ops. is *complete* (CSCO) if join of eigenspace partitions is nondegenerate (i.e., all 1-dim. subspaces). Unique labels are supplied by ordered set of attribute or observable labels, e.g., simultaneous eigenkets $|\lambda,\mu,\ldots\rangle$.

Summary: Lifting from sets to vector spaces

Summary of Lifting Program	Set concept	Vector space concept appropriate for QM
Partition	Set partition $\pi = \{B\}$ of set $U = \cup B$	Direct sum decomposition $\{W_i\}$ of space $V = \Sigma \oplus W_i$
Attribute/observable	Function $f: U \to \mathbb{R}$	Hermitian linear operator $L: V \to V$
Compatible partitions	Partitions π, σ on same set U	Vector space partitions $\{W_i\}$, $\{X_j\}$ with common basis
Compatible attributes	Functions $f, g: U \to \mathbb{R}$ on same domain U	Commuting linear operators $LM = ML$
Partition of attribute	Inverse-image partition $\{f^{-1}(r)\}$ for $f: U \to \mathbb{R}$	Eigenspace partition $\{W_\lambda\}$ for $L: V \to V$
Join compatible attributes	$f^{-1} \vee g^{-1} = \{f^{-1}(r) \cap g^{-1}(s)\}$ for $f, g: U \to \mathbb{R}$	$W_L \vee W_M = \{W_\lambda \cap W_\mu\}$ for $LM = ML$
Complete set of commuting operators	$\vee f_i^{-1}$ is discrete on U for complete set $\{f_i: U \to \mathbb{R}\}$.	$\vee W_{Li}$ is nondegenerate for CSCO $\{L_i: V \to V\}$.

Delifting to "Quantum mechanics" on sets

- Delifting program: creating set versions of QM concepts to have "quantum mechanics" on sets.
- Key step is conceptualizing $\wp(U)$ as $\mathbb{Z}_2^{|U|}$ the |U|-dimensional vector space over 2.
- Vector addition = symmetric difference of sets:

$$S+T = S\cup T - S\cap T.$$

- Example: U = {a,b,c} so U-basis is {a}, {b}, and {c}. Now {a,b}, {b,c}, and {a,b,c} is also a basis since; {a,b}+{a,b,c} = {c}, {b,c}+{c} = {b}, and {a,b}+{b} = {a}. Hence take them as singletons of new basis set U' = {a',b',c'} where {a'} = {a,b}, {b'} = {b,c}, and {c'} = {a,b,c}.
- Ket = row table: $\mathbb{Z}_2^3 \equiv \wp(U) \equiv \wp(U')$

$U=\{a,b,c\}$	$U'=\{a',b',c'\}$
$\{a,b,c\}$	$\{c'\}$
$\{a,b\}$	$\{a'\}$
$\{b,c\}$	$\{b'\}$
$\{a,c\}$	$\{a',b'\}$
$\{a\}$	$\{b',c'\}$
$\{b\}$	$\{a',b',c'\}$
$\{c\}$	$\{a',c'\}$
$\emptyset$	$\emptyset$

"QM" on sets: probabilities

Set Case	Vector space case
Projection $B \cap () : \wp(U) \to \wp(U)$	Projection $P : V \to V$
$f \upharpoonright () = \sum_r r\left(f^{-1}(r) \cap ()\right)$	Herm. $L = \sum_\lambda \lambda P_\lambda$
$\Delta_{B \in \pi} B \cap () = I : \wp(U) \to \wp(U)$	$\sum_\lambda P_\lambda = I$
$\langle S \vert_U T\rangle = \vert S \cap T\vert$ where $S, T \subseteq U$	$\langle \psi \vert \varphi\rangle =$ "overlap" of ψ and φ
$\Vert S\Vert_U = \sqrt{\langle S \vert_U S\rangle} = \sqrt{\vert S\vert}$ where $S \subseteq U$	$\Vert\psi\Vert = \sqrt{\langle\psi\vert\psi\rangle}$
$\Vert S\Vert_U = \sqrt{\sum_{u \in U} \Vert\{u\} \cap S\Vert_U^2}$	$\Vert\psi\Vert = \sqrt{\sum_i \langle v_i\vert\psi\rangle \langle v_i\vert\psi\rangle^*}$
$S \neq \emptyset, \sum_{u \in U} \frac{\Vert\{u\} \cap S\Vert_U^2}{\Vert S\Vert_U^2} = \sum_{u \in S} \frac{1}{\vert S\vert} = 1$	$\vert\psi\rangle \neq 0, \sum_i \frac{\langle v_i\vert\psi\rangle \langle v_i\vert\psi\rangle^*}{\Vert\psi\Vert^2} = 1$
$\Vert S\Vert_U = \sqrt{\sum_r \Vert f^{-1}(r) \cap S\Vert_U^2}$	$\Vert\psi\Vert = \sqrt{\sum_\lambda \Vert P_\lambda(\psi)\Vert^2}$
$S \neq \emptyset, \sum_r \frac{\Vert f^{-1}(r) \cap S\Vert_U^2}{\Vert S\Vert_U^2} = \sum_r \frac{\vert f^{-1}(r) \cap S\vert}{\vert S\vert} = 1$	$\vert\psi\rangle \neq 0, \sum_\lambda \frac{\Vert P_\lambda(\psi)\Vert^2}{\Vert\psi\Vert^2} = 1$
Given S, prob. of r is $\frac{\Vert f^{-1}(r) \cap S\Vert_U^2}{\Vert S\Vert_U^2} = \frac{\vert f^{-1}(r) \cap S\vert}{\vert S\vert}$	Given ψ, prob. of λ is $\frac{\Vert P_\lambda(\psi)\Vert^2}{\Vert\psi\Vert^2}$

Demystifying quantum probabilities using "quantum mechanics" on sets

Distinctions-maker = QM measurement apparatus

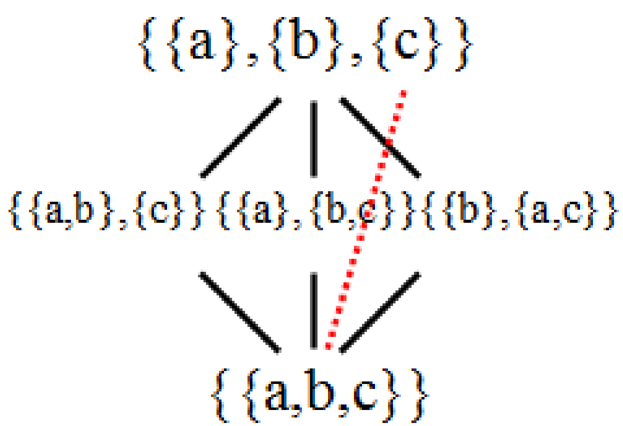
- How are eigenvectors carved out by distinctions?
- In QM, that is a so-called *measurement* of an observable, a distinction-making arrangement to create (NB: not find) a more distinct alternative, i.e., to create objective information.
- If result is degenerate, then a commuting operator is needed to further distinguish the alternatives, and finally a measurement of a CSCO determines eigenvector.

"Measurement" creates objective information

- Pascual Jordan (in 1934) argued that "the electron is forced to a decision. We compel it *to assume a definite position*; previously, in general, it was neither here nor there; it had not yet made its decision for a definite position... . If by another experiment the *velocity* of the electron is being measured, this means: the electron is compelled to decide itself for some exactly defined value of the velocity; and we observe *which* value it has chosen. In such a decision the decision made in the preceding experiment concerning position is completely obliterated." [Quoted in: Jammer, Max 1974. *The Philosophy of Quantum Mechanics: The Interpretations of Quantum Mechanics in Historical Perspective*. New York: John Wiley. p. 161]
- "According to Jordan, every observation is not only a disturbance, it is an incisive encroachment into the field of observation: 'we ourselves produce the results of measurement'." [Ibid.]

Measurement in "QM" on sets

- $U = \{a,b,c\}$ with real-valued attribute $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$.
- Three "eigenspaces": $f^{-1}(1) = \{a\}$, $f^{-1}(2) = \{b\}$, and $f^{-1}(3) = \{c\}$. Take given state $S = \{a,b,c\}$.
- $\Pr(i|S) = |f^{-1}(i) \cap S|/|S| = 1/3$ for $i = 1,2,3$.
- If result was $i = 3$, the "state" resulting from "projective measurement" is $f^{-1}(3) \cap S = \{c\}$.



Indeterminacy principle in "QM" on sets: I

- In previous example of U = {a,b,c} and U' = {a',b',c'} where {a′}={a,b}, {b′}={b,c}, and {c′}={a,b,c}, let f be a real-valued attribute on U and g on U'.
- Don't have operators like $L = \Sigma\lambda P_\lambda$ since only eigenvalues in $\mathbb{Z}_2$ are 0,1, but we do have the projection operators like P_λ, namely $f^{-1}(r)\cap()$ and $g^{-1}(s)\cap()$, so the commutativity properties are stated in terms of those projection operators.
- Let $f = \chi_{\{b,c\}}$ and $g = \chi_{\{a',b'\}}$. The table shows they do not commute.

U	U'	$f\restriction = \{b,c\}\cap()$	$g\restriction = \{a',b'\}\cap()$	$g\restriction f\restriction$	$f\restriction g\restriction$
$\{a,b,c\}$	$\{c'\}$	$\{b,c\}$	0	$\{b,c\}$	0
$\{a,b\}$	$\{a'\}$	$\{b\}$	$\{a'\}=\{a,b\}$	$\{a,c\}$	$\{b\}$
$\{b,c\}$	$\{b'\}$	$\{b,c\}$	$\{b'\}=\{b,c\}$	$\{b,c\}$	$\{b,c\}$
$\{a,c\}$	$\{a',b'\}$	$\{c\}$	$\{a',b'\}=\{a,c\}$	$\{a,b\}$	$\{c\}$
$\{a\}$	$\{b',c'\}$	0	$\{b'\}=\{b,c\}$	0	$\{b,c\}$
$\{b\}$	$\{a',b',c'\}$	$\{b\}$	$\{a',b'\}=\{a,c\}$	$\{a,c\}$	$\{a,c\}$
$\{c\}$	$\{a',c'\}$	$\{c\}$	$\{a'\}=\{a,b\}$	$\{a,b\}$	$\{b\}$
$\emptyset$	$\emptyset$	0	0	0	0

Non-commutativity of the projections $\{b,c\}\cap()$ and $\{a',b'\}\cap()$.

Indeterminacy principle in "QM" on sets: II

- Define that two real-valued attributes $f:U\rightarrow\mathbb{R}$ and $g:U'\rightarrow\mathbb{R}$ "commute" iff their projectors $f^{-1}(r)\cap()$ and $g^{-1}(s)\cap()$ commute.

U	U''	$f\restriction=\{b,c\}\cap()$	$g\restriction=\{a'',b''\}\cap()$	$g\restriction f\restriction$	$f\restriction g\restriction$
$\{a,b,c\}$	$\{a'',b'',c''\}$	$\{b,c\}$	$\{a'',b''\}=\{a,c\}$	$\{c\}$	$\{c\}$
$\{a,b\}$	$\{b'',c''\}$	$\{b\}$	$\{b''\}=\{a\}$	$\emptyset$	$\emptyset$
$\{b,c\}$	$\{a'',c''\}$	$\{b,c\}$	$\{a''\}=\{c\}$	$\{c\}$	$\{c\}$
$\{a,c\}$	$\{a'',b''\}$	$\{c\}$	$\{a'',b''\}=\{a,c\}$	$\{c\}$	$\{c\}$
$\{a\}$	$\{b''\}$	0	$\{b''\}=\{a\}$	$\emptyset$	$\emptyset$
$\{b\}$	$\{c''\}$	$\{b\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{c\}$	$\{a''\}$	$\{c\}$	$\{a''\}=\{c\}$	$\{c\}$	$\{c\}$
$\emptyset$	$\emptyset$	0	0	0	0

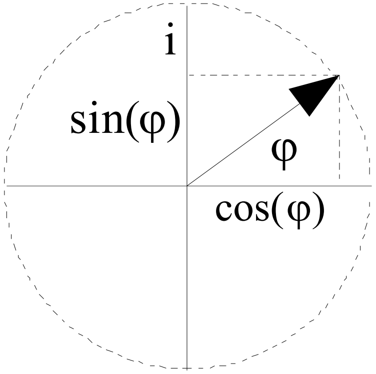
Commuting projection operators $\{b,c\}\cap()$ and $\{a'',b''\}\cap()$.

- Theorem: Linear ops commute iff all their projectors commute iff there exists a basis of simultaneous eigenvectors.
- In this case, simult. basis is $\{a\}=\{b''\}$, $\{b\}=\{c''\}$, and $\{c\}=\{a''\}$.
- This justifies previous defn: f and g compatible iff U = U'.

Obj. Indefiniteness treatment of "wave" equation

- No waves?? What about Schrödinger *wave* equation?
- Measurements make distinctions, so what is the evolution of *closed* quantum system with no interactions that act like measurements?
- Such an evolution would be described by *transformations that hold degree of indefiniteness constant*.
- The degree of indefiniteness or "overlap" between states $|\varphi\rangle$ and $|\psi\rangle$ is given by their inner product $\langle\varphi|\psi\rangle$.
- Hence the transformations of quantum systems that preserve degree of indefiniteness are the ones that preserve inner products, i.e., the *unitary* transformations.

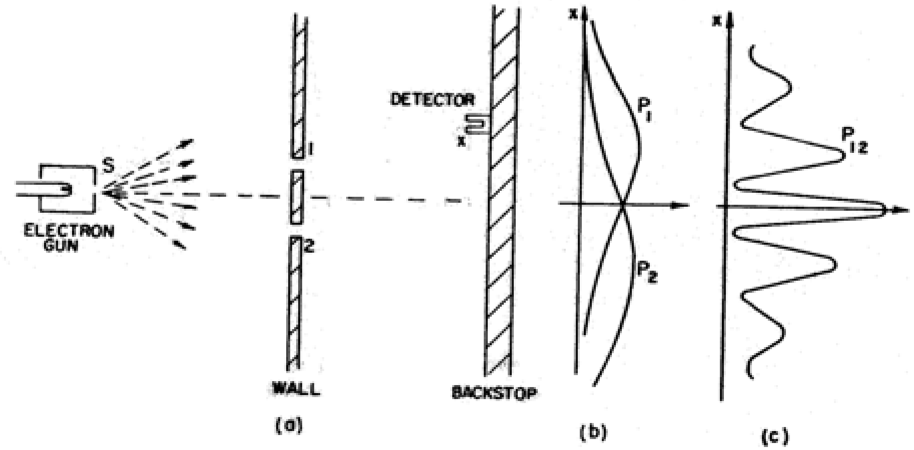
Objective Indefiniteness and "Waves"



- Stone's Theorem gives Schrödinger-style "wave" equation.
- In simplest terms, a unitary transform. describes a rotation in complex plane.
- Rotating unit vector traces out cosine & sine on axes.
- Vector described as function of φ by Euler's formula: $e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$. Complex exponentials & their superpositions are "wave functions" of QM.
- Hence obj. indef. interp. explains the "wave math" (e.g., interference & quantized solutions) when, in fact, there are no actual physical waves.

Two Slit Experiment in QM

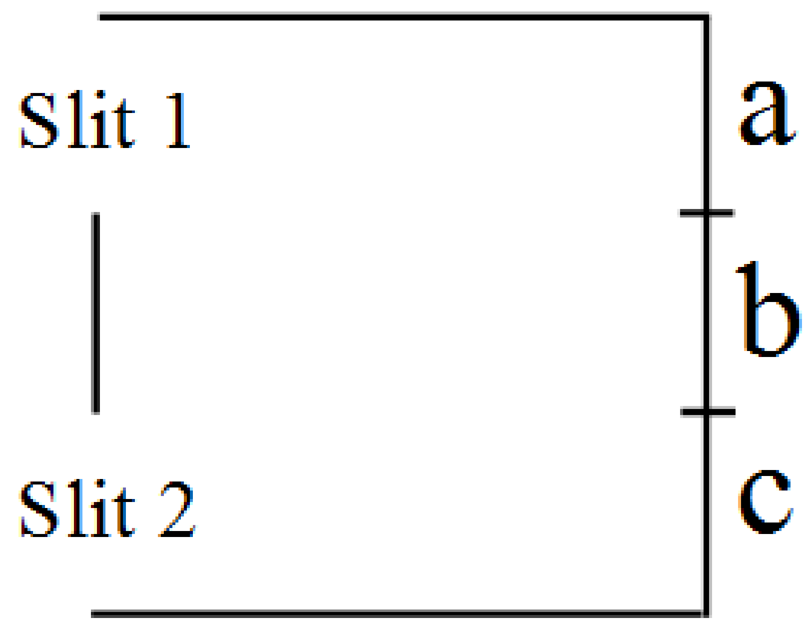
- Double-slit experiment.
  - No distinguishing between slits → "wave-like aspects" appear (i.e., interference) but
  - Distinguish between slits with a measurement (e.g., close a slit or insert detector in a slit) → "wave-like aspects disappear.
- Translation in objective indefiniteness interpretation:
  - No distinctions → "indefiniteness aspects" appear;
  - Make distinctions → "collapse of indefiniteness."



- "If you could, *in principle*, distinguish the alternative *final* states (even though you do not bother to do so), the total, final probability is obtained by calculating the *probability* for each state (not the amplitude) and then adding them together. If you *cannot* distinguish the final states *even in principle*, then the probability amplitudes must be summed before taking the absolute square to find the actual probability." [Feynman et al. Lectures Vol. III, p. 3-9]

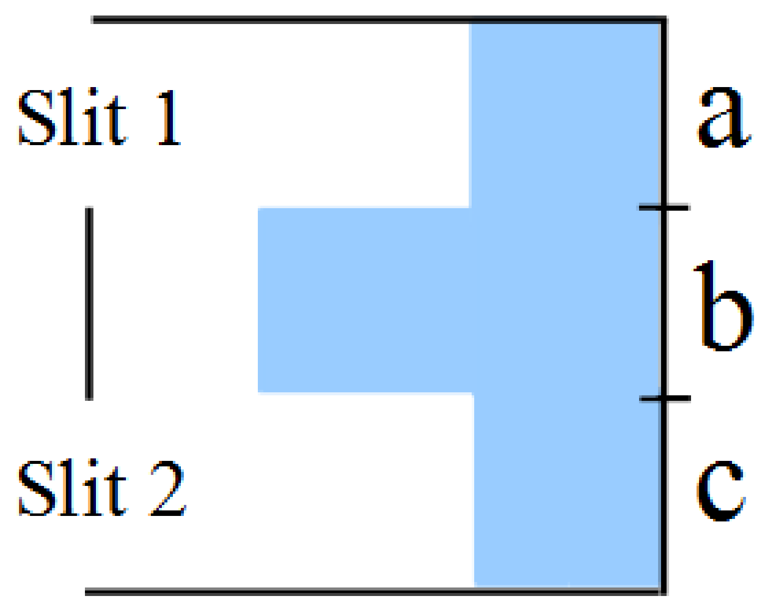
Two Slit Experiment in "QM" on sets: I

- Linear map $A:\mathbb{Z}_2^{|U|} \to \mathbb{Z}_2^{|U|}$ that preserves distinctness is non-singular transformation (no inner product).
- For U={a,b,c}, define A-dynamics by: {a}→{a,b}, {b}→{a,b,c}, and {c}→{b,c}.
- Let basis states {a}, {b}, and {c} represent vertical "positions".
- Two slits on the left, and "particle" traverses box in 1 time period.
- "Particle" hits slits in indefinite state {a,c}.



Two Slit Experiment in "QM" on sets: II

- Case 1: "measurement," i.e., distinctions, at slits.
  - $\Pr(\{a\}|\{a,c\}) = ½$
  - $\Pr(\{c\}|\{a,c\}) = ½$.
- If $\{a\}$, then $\{a\}\to\{a,b\}$, and hits wall: $\Pr(\{a\}|\{a,b\}) = ½ = \Pr(\{b\}|\{a,b\})$.
- If $\{c\}$, then $\{c\}\to\{b,c\}$, and hits wall: $\Pr(\{b\}|\{b,c\}) = ½ = \Pr(\{c\}|\{b,c\})$.
- Thus at wall: $\Pr(\{a\}) = \Pr(\{c\}) = ¼$ and $\Pr(\{b\}) = ½$.



Two Slit Experiment in "QM" on sets: III

- Case 2: no "measurement," i.e., no distinctions, at slits.
- {a,c} evolves linearly:
  - {a}→{a,b} and
  - {c}→{b,c} so that:
  - {a}+{c}={a,c}→{a,b}+{b,c} = {a,c}.
- At the wall, Pr({a}|{a,c}) = ½ = Pr({c}|{a,c}).
- "Interference" cancels {b} in: {a,c}→{a,b}+{b,c} = {a,c}.

Logical entropy measures measurement

- In partition logic, the transition, **0** → **1**, from the blob (indiscrete partition) to the discrete partition turns all indistinctions (i,j) [$i \neq j$] of **0** into distinctions of **1** and logical entropy increases from 0 to $1-\Sigma_i p_i^2$.
- In QM, a (nondegenerate) measurement turns pure-state density matrix ρ to the mixed-state diagonal matrix ρ' with the same diagonal entries p_i:

$$\rho = \begin{bmatrix} p_1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & p_2 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & p_n \end{bmatrix} \underset{\text{Measurement}}{\Rightarrow} \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{bmatrix} = \rho'$$

- Hence the logical entropy $h(\rho) = 1-\mathrm{tr}[\rho^2]$ goes from 0 to $h(\rho') = 1-\Sigma_i p_i^2$.
- For any measurement (degenerate or not), the increase in logical entropy $h(\rho')-h(\rho) = \Sigma_{i\neq j}|\rho_{ij}|^2$ = sum of coherence terms $|\rho_{ij}|^2$ that are zeroed or decohered by measurement.

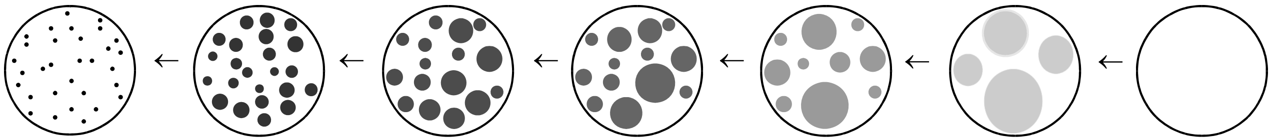
Lifting set products to vector spaces

- Given two set universes U and U', the "composite" universe is their set product $U \times U'$.
- Given two vector spaces H and H' with (orthonormal) bases $\{|i\rangle\}$ and $\{|j\rangle\}$, we get the lifted vector space concept by applying the set concept to the basis sets and then generate the vector space concept.
- The set product of the bases $\{|i\rangle\}$ and $\{|j\rangle\}$ is the set of ordered pairs $\{|i\rangle \otimes |j\rangle\}$ which generate the *tensor product* $H \otimes H'$ (NB: not the direct product $H \times H'$).

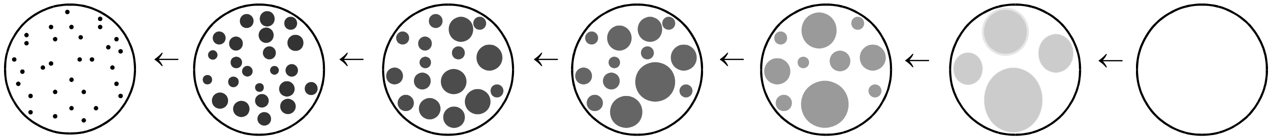
"Deriving" QM by lifting partition math

- Thus by lifting partition math to vector spaces, we essentially get QM: [axioms from Nielsen-Chuang book]
  - Axiom 1: A system is represented by a unit vector in a complex vector space with inner product, i.e., Hilbert space. [lifting program]
  - Axiom 2: Evolution of closed quantum system is described by a unitary transformation. [no-distinctions evolution]
  - Axiom 3: A projective measurement for an observable (Hermitian operator) $M = \Sigma_m m P_m$ (spectral decomp.) on a pure state ρ has outcome m with probability $p_m = \rho_{mm}$ giving mixed state $\rho' = \Sigma_m P_m \rho P_m$. [density matrix treatment of measurement]
  - Axiom 4: The state space of a composite system is the tensor product of the state spaces of component systems. [basis for tensor product = direct product of basis sets]

Objective indefiniteness interpretation



- Starting with universe U as representing common-sense macro-world, there are <u>only two logics</u> to give the creation story:
  - the Boolean logic of subsets, and
  - its dual, partition logic.
- To reach U from the beginning:
  - increase elements in subsets, or
  - increase distinctions in quotient sets.
- *A priori*, either micro-story is possible.
- But the Boolean story is incompatible with QM, so "obvious" idea is try to interpret QM using *the* other story.
- It works! Lifting yields essentially the axioms of QM. (No need for desperate flights of fancy to interpret QM.)
- The result is the *objective indefiniteness interpretation of QM*.



The End

Papers on partition logic and logical information theory available at: www.ellerman.org and www.mathblog.ellerman.org/

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Appendix: Entanglement in "QM" on sets

- Basis principle: direct product $X \times Y$ lifts to the tensor product $V \otimes W$ of vector spaces.
- Subsets of X, Y, and $X \times Y$ correlate (via delifting-lifting) to vectors in V, W, and $V \otimes W$.
- For $S_X \subseteq X$ and $S_Y \subseteq Y$, $S_X \times S_Y \subseteq X \times Y$ is "separated" correlates with $v \in V$ and $w \in W$ giving separated $v \otimes w \in V \otimes W$.
- "Entangled" = Not "separated" subset $S \subseteq X \times Y$.
- Joint prob. dist. $Pr(x,y)$ on $X \times Y$ is *correlated* if $Pr(x,y) \neq Pr(x)Pr(y)$ for marginals $Pr(x)$ and $Pr(y)$.
- Theorem: $S \subseteq X \times Y$ is "entangled" iff equiprobable distribution on S is correlated.

Bell inequality in "QM" on sets: I

- Consider $\mathbb{Z}_2{}^2$ with three incompatible bases U={a,b}, U'={a',b'}, and U"={a",b"} related as in the ket table.
- Given one of the kets as initial state, measurements in each basis have these probs.

kets	U-basis	U'-basis	U''-basis
$\|1\rangle$	$\{a,b\}$	$\{a'\}$	$\{a''\}$
$\|2\rangle$	$\{b\}$	$\{b'\}$	$\{a'',b''\}$
$\|3\rangle$	$\{a\}$	$\{a',b'\}$	$\{b''\}$
$\|4\rangle$	$\emptyset$	$\emptyset$	$\emptyset$

Ket table for $\wp(U) \cong \wp(U') \cong \wp(U'') \cong \mathbb{Z}_2^2$.

Given state \ Outcome of test	a	b	a'	b'	a''	b''
$\{a,b\}=\{a'\}=\{a''\}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	0
$\{b\}=\{b'\}=\{a'',b''\}$	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\{a\}=\{a',b'\}=\{b''\}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1

State-outcome table.

Bell inequality in "QM" on sets: II

- Now form U×U and compute the kets.
- Since {a}={a',b'}={b"} and {b}={b'}={a",b"}, {(a,b)}={a}×{b}={a',b'}×{b'}={(a',b'),(b',b')} ={b"}×{a",b"}={(b",a"),(b",b")}.
- Ket table has 16 rows of these relations but we need the one for an "entangled Bell state":

{(a,a),(b,b)}={(a',a'),(a',b'),(b',a'),(b',b')}+{(b',b')} ={(a',a'),(a',b'),(b',a')}={(a",a"),(a",b"),(b",a")}.

Bell inequality in "QM" on sets: III

- Define prob. dist. Pr(x,y,z) for probability:
  - getting x in U-measurement on left-hand system, &
  - if instead, getting y in U'-meas. on left-hand system, &
  - if instead, getting z in U''-meas. on left-hand system.
- For instance, Pr(a,a',a'') = (1/2)(2/3)(2/3)=2/9.
- Then consider the marginals:
  - Pr(a,a') = Pr(a,a',a'') + Pr(a,a',b'')*
  - Pr(b',b'') = Pr(a,b',b'')* + Pr(b,b',b'')
  - Pr(a,b'') = Pr(a,a',b'')* + Pr(a,b',b'')*.
- Since probs with asterisks in last row occur in other rows and since all probs are non-negative:

$$\Pr(a,a') + \Pr(b',b'') \geq \Pr(a,b'')$$

Bell Inequality

Bell inequality in "QM" on sets: IV

- Consider *independence assumption*: outcome of test on right-hand system independent of test on left-hand system.
- For given initial state, {(a,a),(b,b)}={(a',a'),(a',b'),(b',a')}={(a",a"),(a",b"),(b",a")}, outcomes of initial tests on LH and RH systems have same probabilities.
- Hence prob. distributions Pr(x,y), Pr(y,z), and Pr(x,z) would be the same (under independence) if second variable always referred to test on *right-hand* system.
- With same probs., Bell inequality still holds.

Bell inequality in "QM" on sets: V

- Given state: {(a,a),(b,b)}={(a',a'),(a',b'),(b',a')}={(a",a"),(a",b"),(b",a")}
- To see if independence assumption is compatible with "QM" on sets, we compute the probs.
  - Pr(a,a') gets {a} with prob. ½ but then state of RH system is {a} so prob. of {a'} is ½ (see state-outcome table) so Pr(a,a')=¼.
  - Pr(b',b") gets {b'} with prob. 1/3 but then state of RH system is {a'} and prob. of {b"} is 0, so Pr(b',b")=0.
  - Pr(a,b") gets {a} with prob. ½ but then state of RH system is {a} so prob. of {b"} is 1, so Pr(a,b")=½ .
- Plugging into Bell inequality: Pr(a,a') + Pr(b',b") ≥ Pr(a,b") gives: ¼ + 0 ≥ ½ which is false!
- Hence independence fails & "QM" on sets is "nonlocal."