Double-Entry Accounting: The Mathematical Formulation and Generalization

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Although double-entry accounting has been used in the business world for 5 centuries, the mathematical formulation of the double entry method is almost completely unknown. In this paper, the usual case of scalars is given first and then generalized to the multi-dimensional case of vectors. The success in maintaining the two-sided accounts, debits and credits, the double-entry principle, and the trial balance in both cases provides strong evidence that the formulation correctly captures the double-entry method in mathematical form.

Introduction

Double-entry bookkeeping illustrates one of the most astonishing examples of intellectual insulation between disciplines, in this case, between accounting and mathematics. Double-entry bookkeeping (DEB) was developed during the fifteenth century and was recorded in 1494 as a system by the Italian mathematician Luca Pacioli [1914]. Double-entry book¬keeping has been used for over five centuries in commercial accounting systems. If the mathematical formulation of any field should be well understood, one would think it might be accounting. Remarkably, however, the mathematical formulation of double entry accounting – algebraic operations on ordered pairs of numbers – is largely unknown, particularly in the field of accounting.

The mathematical basis for a precise treatment of DEB was developed in the nineteenth century by William Rowan Hamilton as an abstract mathematical construction using ordered pairs to deal with the complex numbers [Hamilton 1837]. The multiplicative version of this construction is the "group of fractions" which uses ordered pairs of whole numbers (written vertically) to enlarge the system of positive whole numbers to the system of positive fractions containing multiplicative inverses (just reverse the entries in a fraction to get its inverse). The ordered pairs construction that is relevant to conventional DEB is the additive case called the "group of differences."[1] It is used to construct a number system with "additive inverses" by using operations on ordered pairs of positive numbers including zero (unsigned numbers). All that is required to grasp the connection with DEB is to make the identification:

> ordered pairs (horizontally written) of numbers in group of differences construction = two-sided T-accounts of DEB (debits on the left side and credits on the right side).

In view of this identification, the group of differences (or fractions in the multiplicative case) will be called the *Pacioli group.*

In spite of some attention to DEB by mathematicians [e.g., DeMorgan 1869, Cayley 1894, and Kemeny et al. 1962], this connection has not been noted in mathematics (not to mention in

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[1]) See Littlewood 1960, 18; Bourbaki 1974, 20.

accounting) with one perhaps solitary exception. In a semi-popular book, D. E. Littlewood noted the connection:

> The bank associates two totals with each customer's account, the total of moneys credited and the total of moneys withdrawn. The net balance is then regarded as the same if, for example, the credit amounts of £102 and the debit £100, as if the credit were £52 and the debit £50. If the debit exceeds the credit the balance is negative.
>
> This model is adopted in the definition of signed integers. Consider pairs of cardinal numbers (a, b) in which the first number corresponds to the debit, and the second to the credit. [1960, 18]

With this exception, the author has not been able to find a single mathematics book or paper, elementary or advanced, popular or esoteric, which notes that the ordered pairs of the group of differences construction are the T-accounts used in the business world for about five centuries.

Is the Pacioli group the correct formulation of DEB? One acid test of a mathematical formulation of a theory is the question of whether or not it facilitates the generalization of the theory. Normal bookkeeping does not deal with incommensurate physical quantities; everything is expressed in the common units of money. Is there a generalization of DEB to deal with multi-dimensional incommensurates with no common measure of value (e.g., multiple "bottom lines" or environmental accounting)?

In the literature on the mathematics of accounting there was a proposed "solution" to this question, a system of multi-dimensional physical accounting [see Ijiri 1965, 1966, and 1967]. In this system, most of the normal structure of DEB was lost:

- there was no balance sheet equation,
- there were no equity or proprietorship accounts,
- the temporary or nominal accounts could not be closed, and
- the "trial balance" did not balance.

It is common for certain aspects of a theory to be lost in a generalization of the theory. The accounting community had apparently accepted the failure of all these features of DEB as the necessary price to be paid to generalize DEB to incommensurate physical quantities. For example, the systems of "Double-entry

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multidimensional accounting" previously published in the accounting literature [see also Charnes et al. 1976, or Haseman and Whinston 1976] had acquiesced in the absence of the balance-sheet equation.

> For instance, the convenient idea of an accounting identity is lost since the dimensional and metric comparability it assumes is no longer present except under special circumstances. [Ijiri 1967, 333]

Yet when DEB is mathematically formulated using the group of differences, then the generalization to vectors of incommensurate physical quantities is immediate and trivial. *All* of the normal features of DEB – such as the balance-sheet equation, the equity account, the temporary accounts, and the trial balance – are preserved in the generalization [see Ellerman 1982, 1985, 1986]. Thus the "accepted" generalized model of DEB was simply a failed attempt at generalization which had been "received" as a successful generalization that unfortunately had to "sacrifice" certain features of DEB.

Due to this remarkable intellectual insulation between mathematics and accounting, the successful mathematical treatment and generalization of double-entry bookkeeping (first published a quarter-century ago) will take many more years to become known and understood in accounting.

The Pacioli Group

Multi-dimensional accounting is based on the group of differences or Pacioli group construction starting with non-negative vectors. The usual case of accounting can be identified with the special case using one dimensional vectors or scalars. A vector $x = (x_1,...,x_n)$ is *non-negative* if and only if all its components x_i are non-negative (positive or zero). The ordered pairs of non-negative vectors will be called *T-accounts*. The left-hand side (LHS) vector d is the debit entry and the right-hand side (RHS) vector c is the credit entry.

> T-account: [d // c] = [debit vector // credit vector].[2]

The algebraic operations on T-accounts are much like the operations on fractions except that addition is substituted for multiplication. In order to illustrate the additive-multiplicative analogy between T-accounts and fractions, the basic definitions will be developed in parallel columns. For a fraction or "multiplicative T-account", we may take the numerator as the debit entry and the denominator as the credit entry.

The *Pacioli group* P^n consists of the ordered pairs [x // y] of non-negative n-dimensional vectors, with the above definition of addition and equality. The Pacioli group P^n is isomorphic with all of R^n (the set of all n-vectors with positive and negative components) under two isomorphisms: the *debit isomorphism*, which associates [w // x] with w–x, and the *credit isomorphism*, which associates [w // x] with x–w. In order to translate from T-accounts [x // y] back and forth to general vectors z, one needs to specify whether to use the debit or credit isomorphism. This will be done by labeling the T-account as *debit balance* or *credit balance*. Thus if a T-account [x // y] is debit balance, the corresponding vector is x–y, and if it is credit balance, then the corresponding vector is y–x.

The Double-Entry Method: Scalar Case

Given an equation w + ... + x = y + ... + z, it is not possible to change just one term in the equation and have it still hold.

[2]) The double-slash notation was suggested by Pacioli. "At the beginning of each entry, we always provide 'per', because, first, the debtor must be given, and immediately after the creditor, the one separated from the other by two little slanting parallels (virgolette), thus, //,... ." [Pacioli 1914, 43]

Table 1

	Additive Case	Multiplicative Case
Operation on T-accounts	T-accounts add together by adding debits to debits and credits to credits [w // x] + [y // z] = [w+y // x+z].	Fractions multiply together by multiplying numerator times numerator and denominator times denominator (w/x)(y/z) = (wy/xz).
Identity element for operation	The identity element for addition is the zero T-account [0 // 0].	The identity element for multiplication is the unit fraction (1/1).
Equality between two T-accounts.	Given two T-accounts [w // x] and [y // z], the cross-sums are the two vectors obtained by adding the credit entry in one T-account to the debit entry in the other T-account. The equivalence relation between T-accounts is defined by setting two T-accounts equal if their cross-sums are equal: [w // x] = [y // z] if w+z = x+y.	Given two fractions (w/x) and (y/z), the cross-multiples are the two integers obtained by multiplying the numerator of one with the denominator of the other. The equivalence relation between fractions is defined by setting two fractions equal if their cross-multiples are equal: (w/x) = (y/z) if wz = xy.
Inverses	The negative or additive inverse of a T-account is obtained by reversing the debit and credit entries: – [w // x] = [x // w].	The multiplicative inverse of a fraction is obtained by reversing the numerator and denominator: (w/x)–1 = (x/w).
"Disjointness" of two T-accounts	Given two vectors x = (x1,...,xn) and y = (y1,...,yn), let max(x,y) be the vector with the maximum of xi and yi as its ith component, and let min(x,y) be the vector with the minimum of xi and yi as its ith component. Two non-negative vectors x and y are said to be disjoint if min(x,y) = 0.	Given two integers w and x, let lcm(w,x) be the least common multiple of w and x, and let gcd(w,x) be the greatest common divisor of w and x (the largest integer dividing both). Two integers w and x are said to be relatively prime if gcd(w,x) = 1.
"Reduced form" for a T-account	A T-account [x // y] is in reduced form if x and y are disjoint.	A fraction (w/x) is in lowest terms if w and x are relatively prime.
Unique reduced form representation	Every T-account [x // y] has a unique reduced representation [x–min(x,y) // y–min(x,y)].	Each fraction (w/x) has a unique representation in lowest terms (w/gcd(w,x))/(x/gcd(w,x)).
Example	Consider the T-account [(12, 3, 8) // (10, 5, 4)]. The minimum of the debit and credit vectors is (10, 3, 4) so the reduced form representation is [(2, 0, 4) //(0, 2, 0)].	Consider the fraction (28/35). The greatest common divisor of the numerator and denominator is 7 so the fraction in lowest terms is (4/5).

Two or more terms must be changed. The fact that two or more terms (or "accounts") must be changed is *not* the basis for the double-entry method. That mathematical fact is a characteristic of the transaction itself (the changes in the equation), not a characteristic of the method of recording the transaction. The double-entry method is a method of encoding an equation using ordered pairs or T-accounts and using unsigned numbers (non-negative numbers) to record transactions to make changes in the equation. While there is unfortunately considerable confusion about this in the accounting literature, the doubleness of "double-entry" is the two-sidedness of the T-accounts and the mathematical properties that follow (e.g., equal debits and credits in a transaction, and equal debits and credits in the trial balance of the whole set of accounts or ledger).

The alternative to the double entry method is to record a transaction by making a single entry of adding a *signed* (positive or negative) number to each affected account. Two or more accounts in the equation would still always be affected by this alternative method of recording a transaction (since that is a property of the transaction itself, not of the recording method). Such a system is

a complete accounting system to update the balance sheet equation but would have no two-sided T-accounts, no debits or credits, no double entry principle (equal debits and credits in a transaction), and no trial balance of adding debits and credits.
Unfortunately, the phrase "single entry accounting" is also used to denote simply an incomplete accounting "system" (e.g., no equity account) where there is no equation to be updated. But without an equation, that is not an alternative "system" at all. The real choice between the double entry method and the complete single entry method of recording a transaction is the choice between using unsigned ("single-sided") numbers in two-sided accounts or signed ("two-sided") numbers in "single-sided" accounts.

Consider an example of a company with the simplified initial balance sheet equation:

Assets	=	Liabilities	+	Equity
15000	=	10000	+	5000.

Equation 1: Beginning Scalar Balance Sheet

It is customary in accounting (although not mathematically necessary) to move each term or "account" to the side of the equation so that it is preceded by a plus sign. A T-account equal to the zero T-account [0 // 0] is called a *zero-account*. Equations encode as zero-accounts. Each left-hand side (LHS) term x is encoded as a debit-balance T-account [x // 0] and each right-hand side (RHS) term y is encoded as a credit-balance T-account [0 // y]. These T-accounts then would add up to the zero-account [0 // 0]. The balance sheet equation thus encodes as an equation zero-account which, by leaving out the plus signs, becomes the following initial ledger of T-accounts.

Assets	Liabilities	Equity
[15000 // 0]	[0 // 10000]	[0 // 5000]

Equation 2: Beginning Ledger of T-Accounts

Consider three transactions in a productive firm.
1. $1200 of input inventories are used up and charged directly to equity.
2. $1500 of product is produced, sold, and added directly to equity.
3. $800 principal payment is made on a loan.

	Assets	Liabilities	Equity
Original equation zero-account:	[15000 // 0]	[0 // 10000]	[0 // 5000]
+Transaction 1 zero-account:	[0 // 1200]		[1200 // 0]
+Transaction 2 zero-account:	[1500 // 0]		[0 // 1500]
+Transaction 3 zero-account:	[0 // 800]	[800 // 0]	
= Ending equation zero-account:	[16500 // 2000]	[800 // 10000]	[1200 // 6500]
= (in reduced form)	[14500 // 0]	[0 // 9200]	[0 // 5300].

Equation 3: Initial Ledger + Journal = Ending Ledger

Each transaction is then encoded as a transactional zero-account and added to the appropriate terms of the equational zero-account. For instance, the first transaction subtracts 1200 from Assets and subtracts 1200 from Equity. The Assets account is encoded as a LHS or debit-balance account so the subtracting of a number from it would be encoded as adding the T-account [0 // 1200] to it. Equity is encoded as a RHS or credit-balance term so subtracting 1200 from it would be encoded as adding [1200 // 0] to it. The other transactions are encoded in a similar manner.
The initial T-accounts in the ledger add up to the zero account (initial trial balance). Each transaction is encoded as two or more T-accounts that add to the zero-account (double entry principle). Zero added to zero equals zero. Thus adding the transaction zero-accounts to the initial equation zero-account (posting journal to ledger) will yield another equation zero-account (which can be checked by taking another trial balance). Each T-account is then decoded according to how whether it was encoded as debit balance or credit balance to obtain the ending balance sheet equation.

Assets	=	Liabilities	+	Equity
14500	=	9200	+	5300.

Equation 4: Ending Balance Sheet Equation

The Double-Entry Method: General Case

The general case of the double-entry method starts with an equation between sums of n-dimensional vectors. Vector equations are first *encoded* in the Pacioli group constructed from the non-negative n-dimensional vectors. Since the vectors in a T-account must be non-negative, we must first develop a way to separate out the positive and negative components of a vector. The *positive part* of a vector x is $x^+ = \max(x,0)$, the maximum of x and the zero vector [note that "0" is used, depending on the context, to refer to the zero scalar or the zero vector]. The *negative part* of x is $x^- = -\min(x,0)$, the negative of the minimum of x and the zero vector. Both the positive and negative parts of a vector x are non-negative vectors.[3] Every vector x has a "Jordan decomposition" $x = x^+ - x^-$. The two isomorphisms that map vectors to T-accounts of non-negative vectors are the debit isomorphism that maps x to the T-account [x^+ // x^-] and the credit isomorphism that maps x to [x^- // x^+].

We must first develop a way to separate out the positive and negative components of a vector

Given any equation in Rn, w + ... + x = y + ... + z, each left-hand side (LHS) vector x is encoded via the debit isomorphism as a debit-balance T-account [x^+ // x^-] and each right-hand side (RHS) vector y is encoded via the credit isomorphism as a credit-balance T-account [y^- // y^+]. Then the original equation holds if and only the sum of the encoded

T-accounts is a zero-account:

$$w + \ldots + x = y + \ldots + z$$
if and only if
$$[w^{+} // w^{-}] + \ldots + [x^{+} // x^{-}] + [y^{-} // y^{+}] + \ldots + [z^{-} // z^{+}]$$ is a zero-account.

Equation 5: Encoding an Equation as an Equation Zero-Account

Given the equation, the sum of the encoded T-accounts is the *equation zero-account* of the equation. Since only plus signs can appear between the T-accounts in an equational zero-account, the plus signs can be left implicit. The listing of the T-accounts in an equational zero-account (without the plus signs) is the *ledger*.

Changes in the various terms or "accounts" in the beginning equation are recorded as *transactions*. Transactions must be recorded as valid algebraic operations which transform equations into equations. Since equations encode as zero-accounts, a valid algebraic operation would transform zero-accounts into zero-accounts. There is only one such operation in the Pacioli group: add on a zero-account. Zero plus zero equals zero. The zero-accounts representing transactions are called *transaction zero-accounts*. The listing of the transactional zero-accounts is the *journal*.

A series of valid additive operations on a vector equation can then be presented in the following standard scheme:

Beginning Equation Zero-Account
+ Transaction Zero-Accounts
= Ending Equation Zero-Account

or, in more conventional terminology,

Beginning Ledger
+ Journal
= Ending Ledger.

The process of adding the transaction zero-accounts to the initial ledger to obtain the ledger at the end of the accounting period is called *posting the journal to the ledger*. The fact that a transaction zero-account is equal to [0 // 0] is traditionally expressed as the *double-entry principle* that transactions are recorded with equal debits and credits. The summing of the debit and credit sides of what should be an equation zero-account to check that it is indeed a zero-account is traditionally called the *trial balance*. All those features from scalar case of DEB carry over to the general vector case.

At the end of the cycle, the ending equational zero-account is decoded to obtain the equation that results from the algebraic operations represented in the transactions. The T-accounts in an equational zero-account can be arbitrarily partitioned into two sets: DB (debit balance) and CB (credit balance). T-accounts [w // x] in DB are decoded as w–x on the left side of the equation, and T-accounts [w // x] in CB are decoded as x–w on the right side of the equation. Given a zero-account, this algorithm yields an equation. In an accounting application, the T-accounts in the final equation zero-account would be partitioned into sets DB and CB according to the side of the initial equation from which they were encoded.

Consider the following initial vector equation:

(6, –3, 10) + (–2, 5, –2) = (4, 2, 8).
Equation 6: Sample Vector Equation to be Encoded

It encodes as the equation zero-account

[(6, 0, 10) // (0, 3, 0)] + [(0, 5, 0) // (2, 0, 2)] + [(0, 0, 0) // (4, 2, 8)].
Equation 7: Equation Encoded as a Zero T-Account

Suppose that the transaction would subtract the vector (–2, –9, 1) from the first vector on the LHS and from the vector on the RHS side of the original equation to obtain the ending equation:

(8, 6, 9) + (–2, 5, –2) = (6, 11, 7).
Equation 8: Ending Vector Equation

To perform this operation using the double-entry method, the subtracting of the vector (–2, –9, 1) from the first LHS term is encoded using the credit isomorphism to get [(2,9,0)//(0,0,1)] which is added to the first LHS or debit-balance term in the T-account version of the original equation. In more traditional terminology, we would say that (–2, –9, 1) is "credited" to that debit-balance account. For the subtraction from the RHS term, the vector is encoded using the debit isomorphism to obtain [(0,0,1)//(2,9,0)] and added to the credit-balance T-account version of the RHS term. That is, (–2, –9, 1) is "debited" to that credit-balance account. This yields another equational zero-account:

Original Equation zero-account:
[(6,0,10)//(0,3,0)] + [(0,5,0)//(2,0,2)] + [(0,0, 0)//(4,2,8)]
+ Transaction zero-account:
[(2,9,0)//(0,0,1)] + [(0,0,1)//(2,9,0)]
Ending equation zero-account:
[(8,9,10)//(0,3,1)] + [(0,5,0)//(2,0,2)] + [(0,0,1)//(6,11,8)].

Equation 9: Beginning Ledger + Journal = Ending Ledger

After a number of such transactions, the ending equation zero-account is then decoded to obtain an equation back in Rn. In this case, let the first two T-accounts be debit-balance and the third one credit-balance (as they were originally encoded). Then the ending equational zero-account decodes as the vector equation

(8, 6, 9) + (–2, 5, –2) = (6, 11, 7).
Equation 10: Decoded Ending Equation

In the scalar case, a T-account will always have a reduced form either as [d // 0] or [0 // c] so that adding [d // 0] to an account (a term in the equational zero-account) can be described as "debiting d to the account" and similarly for "crediting c to the account." For vector T-accounts, the reduced form of a T-account does not necessarily have the zero vector on one side or the other. In the case above, the reduced form of the T-account encoding of (−2, −9, 1) would be "mixed." The "debit" takes the form of adding the T-account [(0,0,1)//(2,9,0)] obtained using the debit isomorphism to a term, and the "credit" takes the form of adding the inverse [(2,9,0)//(0,0,1)] obtained by the credit isomorphism to another term.

Concluding Remarks

"Mathematics is the study of analogies between analogies."[Rota 1997, 214] The mathematical analysis of double-entry bookkeeping starts with the analogy between the ordered pairs construction of additive inverses and T-accounts. That allows the concrete procedures of DEB to be reproduced abstractly using the group of differences construction, and then to be extended from ordered pairs of non-negative scalars to ordered pairs of non-negative vectors to obtain the system of n-dimensional DEB. The immediate and straightforward generalization of the ordered pairs treatment to n-dimensional vectors and to fractions – with the main structure and principles of DEB preserved – supports the thesis that this treatment captures the mathematical essence of the double-entry method.

This mathematical treatment of DEB serves another purpose – to help clear up the remarkable confusion in the accounting literature about the "doubleness" that is characteristic of DEB. The conventional wisdom is still that the doubleness refers to the fact that each transaction must affect two or more accounts (see any textbook) when that is a characteristic of the transaction itself, not the recording method. Any complete accounting system, such as the double entry method using "single-sided" numbers (no minus signs) with double-sided accounts, or a complete single-entry system using "double-sided" numbers (positive or negative) in single-sided accounts, must change two or more accounts in recording a transaction.

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